Generalizing Top Trading Cycles for Housing Markets with Fractional Endowments

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Abstract. The housing market setting constitutes a fundamental model of exchange economies of goods. In most of the work concerning housing markets, it is assumed that agents own and are allocated discrete houses. The drawback of this assumption is that it does not cater for randomized assignments or allocation of time-shares. Recently, house allocation with fractional endowment of houses was considered by Athanassoglou and Sethuraman (2011) who posed the open problem of generalizing Gale's Top Trading Cycles (TTC) algorithm to the case of housing markets with fractional endowments. In this paper, we address the problem and present a generalization of TTC called FTTC that is polynomial-time as well as core stable and Pareto optimal with respect to stochastic dominance even if there are indifferences in the preferences. For the standard setting in which each agent owns one discrete house, FTTC coincides with a state of the art strategyproof mechanism for housing markets with discrete endowments and weak preferences. We show that FTTC satisfies a maximal set of desirable properties by proving two impossibility theorems. Firstly, we prove that with respect to stochastic dominance, core stability and no justified envy are incompatible. Secondly, we prove that there exists no individual rational, Pareto optimal and weak strategyproof mechanism, thereby answering another open problem posed by Athanassoglou and Sethuraman (2011). The second impossibility implies a number of results in the literature.

1 Introduction

The housing market is a fundamental model of exchange economy of goods. It been used to model online barter markets and nation-wide kidney markets [21, 25]. The housing market (also called the *Shapley-Scarf* market) consists of a set of agents each of whom owns a house and has preferences over the set of houses. The goal is to redistribute the houses among the agents in an efficient and stable manner. The desirable properties include the following ones: *Pareto optimality* (there exists no other assignment which each agent weakly prefers and at least one agent strictly prefers); *individual rationality* (the resultant allocation is at least as preferred by each agent as his endowment); and *core stability* (there exists no subset of agents who could have redistributed their endowments among themselves so as to get a more preferred outcome than the resultant assignment).

Shapley and Scarf [23] showed that for housing markets with strict preferences, an elegant mechanism called *Gale's Top Trading Cycles (TTC)* (that is based on multi-way exchanges of houses between agents) is strategyproof and finds an allocation that is in the core [23, 15].[1] Along with the Deferred Acceptance Algorithm, TTC has provided the foundations for many of the developments in matching market design [16, 25]. The Shapley-Scarf market has been used to model important real-world problems for allocation of human organs and seats at public schools. Since the formalization of TTC, considerable work has been done to extend and generalize TTC for more general domains that allow indifference in preferences [1, 12, 6, 18, 22] or multiple units in endowment [11, 14, 26, 28].

Despite recent progress on house allocation and housing market mechanisms, the general assumption has remained that agents cannot own or be allocated fractions of houses. The disadvantage of this assumption is that it does not model various cases where agents have fractional endowments or when agents can share houses. This is especially the case when agents have the right to use different facilities for different fractions of the time and fractional allocation of resources is helpful is obtaining more equitable outcomes. Fractional allocation of houses can also be interpreted as the relative right of an agent over an house [2]. Finally, fractional allocations can be used to model randomized allocation of indivisible resources where agents exchange probabilities of getting particular houses. Hence allocation of houses under fractional endowments generalizes a number of well-studied house allocation models. If there are no fractional endowments but fractional allocation is possible, then we end up in the random assignment model [10]. If the endowments are discrete, then we recover the housing market model. If the endowment matrix is a permutation matrix, we end up in the basic house trading model of Shapley and Scarf [23] in which each agent owns a distinct house [2].

Although important mechanisms have been proposed for house allocation, random assignment, and housing markets, it has not been clear how to generalize TTC for housing markets with fractional endowments so that the properties enjoyed by TTC such as core stability are still satisfied. This fundamental problem was raised by Athanassoglou and Sethuraman [2] who were the first to examine house allocation in which agents are endowed with fractions of houses. They presented an algorithm that returns an assignment that is individually rational and satisfies a fairness concept called no justified envy. Athanassoglou and Sethuraman [Page 512, 2] posed an open problem whether it is possible to extend TTC to handle fractional endowments. They also highlighted the core of the fractional housing market as an interesting topic: *"The most appropriate way to define the core is not apparent; our preliminary investigation suggests mostly*

[1] The seminal paper of Shapley and Scarf [23] was referenced prominently in the scientific background document of the *2012 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel* given to Lloyd Shapley and Alvin Roth.

negative results, but much remains to be done here. Finally, an interesting (and challenging) open question is to generalize the TTC mechanism to this setting." In the same paper, Athanassoglou and Sethuraman [2] also asked another question: *"In light of this and other impossibility results, a natural question to ask is whether there exists a mechanism that is individually rational, ordinally efficient, and weakly strategyproof."*

In this paper, we answer the two open questions posed by Athanassoglou and Sethuraman [2]. We use the SD (stochastic dominance) relation to define the core and present an SD-core stable algorithm that generalizes TTC. We also refer to individual rationality as SD-IR, ordinal efficiency as SD-efficiency, and weak strategyproof as weak SD-strategyproof. We prove that the three properties are incompatible.

Contributions In this paper, we propose a polynomial-time algorithm called *FTTC (Fractional Top Trading Cycle)* which is designed for housing markets with fractional endowments. With respect to the stochastic dominance relation, we prove that FTTC is individually rational, Pareto optimal and core-stable. In contrast, the previously proposed controlled-consuming (CC) algorithm of Athanassoglou and Sethuraman [2] for fractional housing markets is not SD-core stable.

We show that FTTC satisfies a maximal set of desirable properties by proving two impossibility theorems. Firstly, we show that core stability and no justified envy are incompatible. Secondly, we prove that any mechanism that is individually rational and Pareto optimal cannot be weak strategyproof which answers the second open problem posed by Athanassoglou and Sethuraman [2]. The impossibility result also implies prior impossibility results in the literature [2, 29]. We then prove that although FTTC is not weak SD-strategyproof, checking whether there exists a manipulation for a given agent that is SD-preferred over the truthful outcome is an NP-hard problem.

Even though FTTC is designed for fractional allocations and fractional endowments, FTTC coincides with the state of the art mechanisms for discrete house allocation and housing markets (see Table 1). In this way, we unify and generalize the previous mechanisms in the literature. For discrete endowments, all our positive results with respect to SD relation translate to positive results with respect to the responsive set extension.

Domain restriction	Mechanism
unrestricted	FTTC
strict preferences, discrete and single endowments	TTC
discrete and single endowments	Plaxton's mechanism [18]
strict preferences, discrete endowments	ATTC [11].
no endowments	Serial Dictatorship

Table 1. Equivalence of FTTC with known mechanisms on restricted domains.

2 Related Work

The fractional assignment model with endowments was first examined by Yilmaz [29] and Athanassoglou and Sethuraman [2]. Yilmaz [29] presented an interesting generalization of the *probabilistic serial* random assignment mechanism of Bogomolnaia and Moulin [10] to the setting where the houses are endowments of the agents. Athanassoglou and Sethuraman [2] generalized the algorithm of Yilmaz [29] to the case where the endowments may be fractional and the preferences over individual houses need not be strict.

Apart from the work of Yilmaz [29] and Athanassoglou and Sethuraman [2], the remaining literature focuses on discrete allocation of indivisible resources that does not consider house allocation with fractional endowments. Discrete housing markets with indifferences have been analyzed in a number of recent papers [1, 12, 6, 18, 19, 22]. Alcalde-Unzu and Molis [1] and Jaramillo and Manjunath [12] proposed desirable mechanisms (called TTAS and TCR respectively) for housing markets with indifferences. Aziz and de Keijzer [6] outlined a simple class of mechanism called GATTC which encapsulate TCR and TTAS and satisfy many desirable properties of the two mechanisms. In the model we consider, an agent can be allocated more than one house or units from different houses [26, 17, 14]. Recently, Biró et al. [9] examined a discrete exchange setting in which agents have strict preferences over objects but there can be multiple copies of objects. They study the conditions under which strategyproofness can be acheived. Our focus is different and our model is general in at least two respects: we allow subjective indifference in the preferences and also allow fractional endowments.

Allocation of *discrete* multiple objects to agents has been considered before [11, 14, 17, 24, 26, 28]. Papai [17] assumed strict preferences over bundles of objects which does not allow the flexibility to incorporate indifferences. Konishi et al. [14] and Sonoda et al. [26] also considered housing markets with multiple goods but mainly presented negative results such as the emptiness of the core for general preferences over sets of houses. Similarly, Todo et al. [28] showed that under the lexicographic preference domain, there exists no exchange rule that satisfies strategyproofness and Pareto efficiency. Fujita et al. [11] studied a natural extension of TTC in which each agent has *strict* preferences over houses and preferences over sets of houses are derived via the lexicographic set extension. Each agent is divided into subagents with each agent owning exactly one discrete house and then the standard TTC is applied to the market with the subagents. In the setting we consider, agents may be indifferent between houses and there may be different units of houses in the markets and the allocation need not be discrete. Dividing the fractional houses into discrete houses may result in an exponential blowup in the size of the market. Sönmez [24] examined general exchange and matching models and showed that in a large class of such models, there exists a Pareto efficient, individually rational, and strategyproof solution only if all allocations in the core are Pareto indifferent for all problems. When exchanging multiple indivisible goods, Atlamaz and Klaus [3] discussed how agents may have an incentive to hide some endowments.

3 Preliminaries

3.1 Model

Consider a market with set of agents $N = \{1, \dots, n\}$ and a set of houses $H = \{h_1, \dots, h_m\}$. Each agent has complete and transitive preferences $\succsim_i$ over the houses and $\succsim = (\succsim_1, \dots, \succsim_n)$ is the preference profile of the agents. Agents may be indifferent among houses. We denote $\succsim_i$: $E_i^1, \dots, E_i^{k_i}$ for each agent i with equivalence classes in decreasing order of preference. Thus, each set E_i^j is a maximal equivalence class of houses among which agent i is indifferent, and k_i is the number of equivalence classes of agent i. An agent has *dichotomous preferences* if he considers each house as either acceptable or unacceptable and is completely indifferent between unacceptable houses and also indifferent between acceptable houses.

Each agent i is endowed with allocation $e(i)$ where $e(i)(h_j)$ units of house h_j given to agent i. The quadruple $(N, H, \succsim, e)$ is a *housing market with fractional endowments*. Note that in the basic housing market, each agent is endowed with and is allocated one house and the endowments are discrete: $n = m$, $e(i)(h_j) \in \{0, 1\}$ and $\sum_{h \in H} e(i)(h) = 1$ for all $i \in N$ and $\sum_{i \in N} e(i)(h) = 1$ for all $h \in H$. When allocations are discrete we will also abuse notation and denote $e(i)$ as a set.

A *fractional assignment* is an $n \times m$ matrix $[x(i)(h_j)]_{1 \le i \le n, 1 \le j \le m}$ such that for all $i \in N$, and $h \in H$, $\sum_{i \in N} x(i)(h) = \sum_{i \in N} e(i)(h)$. The value $x(i)(h_j)$ is the fraction or units of house h_j that agent i gets. We will use fraction or unit interchangeably since we do not assume that exactly one unit of each house is in the market. Each row $x(i) = (x(i)(h_1), \dots, x(i)(h_m))$ represents the *allocation* of agent i. Given two allocations $x(i)$ and $x(j)$, $x(i) + x(j)$ is the point-wise sum of the allocations $x(i)$ and $x(j)$. If $\sum_{i \in N} x(i)(h) = 1$ for each $h \in H$, a fractional assignment can also be interpreted as a random assignment where $x(i)(h_j)$ is the probability of agent i getting house h_j. Note that endowment e itself can be considered as the initial assignment of houses to the agents with $e(i)$ being the initial allocation of agent $i \in N$. A *fractional housing market* mechanism is a function that takes as input $(N, H, e, \succsim)$ and returns an assignment or vector of allocations $(x(1), \dots, x(n)$ such that $\sum_{i \in N} x(i) = \sum_{i \in N} e(i)$. We do not require in general that $\sum_{i \in N} e(i)(h)$ or $\sum_{h \in H} e(i)(h)$ are integers.

Example 1 (Discrete Housing Market and TTC).

Consider the following housing market $(N', H, \succsim, e)$ where $N = \{1, 2, 3\}$, $H = \{a, b, c\}$, The endowment assignment can be represented as follows:

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$
\begin{aligned}
1:&\quad b,c,a\\
2:&\quad c,a,b\\
3:&\quad a,b,c
\end{aligned}
$$

Since for all $i \in N$, $h \in H$, $e(i)(h) \in \{0,1\}$, $\sum_{h\in H} e(i)(h) = 1$, and $\sum_{i\in N} e(i)(h) = 1$, the housing market is equivalent to the basic housing market. The outcome of running the TTC algorithm is that each agent gets its most preferred house:

$$TTC(N, H, \succsim, e) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

3.2 Properties of allocations and mechanisms

Before defining various stability and efficiency properties, we need to reason about agents' preferences over allocations. A standard method to compare random allocations is to use the *SD (stochastic dominance)* relation. Given two random assignments x and y, $x(i) \succsim_i^{SD} y(i)$ i.e., an agent i *SD prefers* allocation $x(i)$ to $y(i)$ if $\forall h \in H : \sum_{h_j \in \{h_k : h_k \succsim_i h\}} x(i)(h_j) \geq \sum_{h_j \in \{h_k : h_k \succsim_i h\}} y(i)(h_j)$.

The SD relation is not complete.

We define normative properties of allocations as well as mechanisms.

- *SD-efficiency*: an assignment x is *SD-efficient* if there exists no other assignment y such that $y(i) \succsim_i^{SD} x(i)$ for all $i \in N$ and $y(i) \succ_i^{SD} x(i)$ for some $i \in N$.
- *SD-core*: an assignment x is in *SD-core* if there exists no other coalition $S \subseteq N$ and an assignment y for agents in S such that $\sum_{i\in S} y(i) = \sum_{i\in S} e(i)$ and $y(i) \succ_i^{SD} x(i)$ for all $i \in S$.
- *SD strict core*: an assignment x is in *SD- strict core* if there exists no other coalition $S \subseteq N$ and an allocation y on S such that $\sum_{i\in S} y(i) = \sum_{i\in S} e(i)$, $y(i) \succsim_i^{SD} x(i)$ for all $i \in S$ and $y(i) \succ_i^{SD} x(i)$ for some $i \in S$.
- *SD individually rational*: an assignment x is *SD-individually rational* if $x(i) \succsim_i^{SD} e(i)$.
- A mechanism f is *SD-manipulable* iff there exists an agent $i \in N$ and preference profiles $\succsim$ and $\succsim'$ with $\succsim_j = \succsim_j'$ for all $j \neq i$ such that $f(\succsim') \succ_i^{SD} f(\succsim)$. A mechanism is *weakly SD-strategyproof* iff it is not SD-manipulable, it is *SD-strategyproof* iff $f(\succsim) \succsim_i^{SD} f(\succsim')$ for all $\succsim$ and $\succsim'$ with $\succsim_j = \succsim_j'$ for all $j \neq i$.

Remark 1. For any lottery extension, SD strict core stability implies SD core stability. Moreover, SD strict core implies SD-efficiency.

When each agent owns exactly one discrete house and each assignment results with each agent owning one house, then individual rationality and core with respect to SD coincide with the standard notions as used in [1, 6, 12]. SD-individually rationality is equivalent to individual rationality; SD-efficiency is equivalent to Pareto optimality; SD-core is equivalent to core; SD-strict core is equivalent to strict core; SD-strategyproofness, weak SD-strategyproofness and DL-strategyproofness are equivalent to strategyproofness.

In the absence of endowments, the concept is envy-freeness can be easily defined: $x(i) \succsim_i^{SD} x(j)$ for all $i, j \in N$. However, envy-freeness needs to be redefined when taking into account endowments of agents. Athanassoglou and Sethuraman [2] defined *no justified envy (NJE)* as follows. An agent i has *justified envy* towards agent j if $x(j) \succ_i^{SD} x(i)$ and $x(i) \succsim_j^{SD} e(j)$. An assignment satisfies *no justified envy (NJE)* if no agent $i \in N$ has justified envy towards some other agent $j \in N$. The notion is weaker than the NJE notion defined by Yilmaz [29].

4 FTTC: Fractional Top Trading Cycles Algorithm

We present FTTC (Fractional Top Trading Cycles) that is an algorithm based on multi-way exchanges of fractions of houses. In contrast to TTC for the single unit discrete housing markets, multiple houses can be owned by an agent. Hence, we divide each agent $i \in N$ into m subagents with each subagent i_h corresponding to one of the house $h \in H$. A subagent i_h owns or is allocated a fraction of h on behalf of i. During the running of FTTC each subagent i_h can only own units of h and not other houses. Each subagent i_h has the same preferences as agent i. FTTC is implemented by maintaining a graph where the set of vertices are associated with subagents and houses. An edge from a subagent to a house indicates the house is a maximally preferred among the houses in the graph by the subagent, an edge from a house to a subagent indicates that the subagent has a non-zero fractional ownership of that house. The graph corresponds to the part of the assignment that is not yet finalized. During the running of FTTC, the graph and hence the current assignment is modified. Those houses and subagents that are removed from their graph, their assignment is already finalized. The algorithm is outlined as Algorithm 1. Where the context is clear, we will refer to a subagent simply as the agent he is representing. For example, if house h is removed from the graph, the allocation of h to different (sub)agents has been fixed.

A *trade* is specified by a cycle of the form $h_1, 1, h_2, 2 \ldots, h_k, k, h_1$ such that each agent i transfers α units of his house h_i to agent $(i+1) \mod (k)$ where $\alpha \leq e(i)(h_i)$. If the trade includes an agent i such that $h_i \notin \max_{\succsim_i}(H)$ but $h_{(i+1) \mod (k)}$, then the trade is referred to as a *good* cycle. A cycle that is not good will be referred to as *non-good*. In the housing market, if an agent i owns a house h such that $h \notin \max_{\succsim_i}(H)$, then we call i an *attractor*. The reason for calling such an agent an attractor is that we will want other houses and agents to form a path towards such an agent.

Algorithm 1 FTTC algorithm for house markets with fractional endowments.

Input: $(N, H, \succsim, e)$

Output: An SD-individually rational, SD-efficient, and SD-core stable allocation.

$x \longleftarrow e$ {We maintain assignment x}

Based on $(N, H, \succsim, e)$, construct a graph $G = (N^* \cup H, E, w)$ where

- $N^* \longleftarrow \{i_h : i \in N, h \in H\}$ {$N^*(G)$ denotes the set of subagents in the graph}
- $(i_h, h') \in E$ iff $h' \in \max_{\succsim_i}(H(G))$ and $h \not\succ_i \max_{\succsim_i}(H(G))$
- $(h, i_h) \in E$ iff $x(i)(h) > 0$; edge weight $w(h, i) \longleftarrow x(i)(h_j)$

Consider a tie-breaking priority ranking L_H over the houses and L_N over the agents.

Maintain set of *attractor* subagents (a subagent i_h such that $x(i)(h) > 0$ and $h \notin \max_{\succsim_i}(H(G))$).

Maintain $d(v)$ — the shortest distance of vertex v to an attractor. Thus $d(i) = 0$ and $d(h) = 1$ if i_h is an attractor.

{We will next consider absorbing sets (of unweighted version of G) which can be found via Tarjan's algorithm [27]. Among absorbing sets, we consider non-good absorbing sets that do not contain a good cycle (i.e., abosorbing sets in which no subagent points to a house strictly more preferred to a house it currently owns)}

while $N^*(G) \neq \emptyset$ **do**

  while G has at least one non-good absorbing set **do**

    In each non-good absorbing set, delete those houses and subagents from G (since allocation for the agents is now completely fixed for the houses in the absorbing set)

    Readjust the graph (subagents now have arcs to the most preferred houses in the modified graph). Remove any subagent i_h if h has been removed from G.

  end while

  Identify each (disjoint) cycle C by making each vertex in G point to exactly one other vertex in the following manner:

  - Among the owners of h, make h point to the highest ranked owner with the minimum distance to an attractor.

$$\text{Next}(h) \longleftarrow \max_{L_N}(\arg\min_{d(\cdot)}(x_h(N^*(G)))).$$

  Let $P(i_h)$ be the set of houses pointing to subagent i_h.

  - Make each subagent i_h point to the highest ranked house with the minimum distance to an attractor.

$$\text{Next}(i) \longleftarrow \max_{L_H}(\arg\min_{d(\cdot)}(\max_{\succsim_i}(H(G)) \setminus P(i_h))).$$

  {Note that each subagent of the same agent is made to point to the same house.}

  For each cycle C, compute $\alpha = \min_{(h', i_h) \in C} w(h', i)$.

  for each $h \in C$ **do**

    In cycle C, $j_{h'}$ points to h which points to subagent i_h.

$$w((h), i_h) \longleftarrow w((h), i_h) - \alpha.$$

$$w((h), j_h) \longleftarrow w((h), i_h) + \alpha.$$

    {although $j_{h'}$ was in the cycle and pointing to h, when the exchange happens, the fraction α of h is given to subagent j_h because only subagent j_h that corresponds to house h keeps ownership of h}.

  end for

  Re-adjust the graph (this includes deleting any edge (h, i) if $w(h, i) = x(i)(h) = 0$).

  Readjust set of attractors if needed.

end while

In FTTC, based on the graph, we can identify *absorbing sets* of vertices where an absorbing set is a strongly connected component of a graph with no arcs going outside the component. Those absorbing sets that do not contain a good cycle are deleted from the graph. The deletion can be interpreted as finalizing the allocation of the houses in the absorbing set.[2] After deleting non-good absorbing sets (if any), we implement trades of houses among the agents. In order to implement trades, we make each vertex point to exactly one other vertex. Like the mechanisms by Jaramillo and Manjunath [12] and Plaxton [18], a (sub)agent points to a house which has the shortest path to an attractor. However, since a house can be owned fractionally by various agents, we need to induce a second tie-breaking order L_N over the agents so that if a house has multiple (sub)agents pointing to it, it points to the one with the highest priority. Since each vertex has outdegree one, a cycle exists. Furthermore, since vertices point in the direction of attractors, we identify good cycles. We can then trade the maximum possible units of houses in the cycle. In each trade, agents involved in the trade replace some units of a house by equal number of units of a house that is at least as preferred. FTTC is described formally as Algorithm 1. An important observation is that when the houses of an absorbing set are removed then all houses that are equally preferred by at least one agent are removed as well.

Example 2 (Illustration of FTTC). Consider the following housing market $(N', H, \succsim, e)$ where $N = \{1, 2, 3\}$, $H = \{a, b, c\}$, The endowment assignment can be represented as follows:

$$e = \begin{pmatrix} 0 & 0.99 & 0.01 \\ 0.99 & 0 & 0.01 \\ 0.01 & 0.01 & 0.98 \end{pmatrix}.$$

$$\begin{aligned} 1: &\quad a, c, b \\ 2: &\quad b, a, c \\ 3: &\quad b, a, c \end{aligned}$$

Firstly, FTTC forms subagents: $N^* = \{1_a, 1_b, 1_c, 2_a, 2_b, 2_c, 3_a, 3_b, 3_c\}$. Suppose that $L_N = 1, 2, 3$ and $L_H = a, b, c$. Then, the first good cycle encountered is $a, 2_a, b, 1_b$ which leads to an exchange of 0.99 units of houses a and b between agent 1 and 2.

$$x = \begin{pmatrix} 0.99 & 0 & 0.01 \\ 0 & 0.99 & 0.01 \\ 0.01 & 0.01 & 0.98 \end{pmatrix}.$$

[2] Just like the GATTC mechanism [6], absorbing sets are computed but instead of deleting paired-symmetric (sets in which each pair of vertices point at each other) absorbing sets, those absorbing sets are deleted that do not contain a good cycle. Note that in the case of GATTC, both conditions are equivalent but in the case of fractional endowments, one also needs to consider how many units of a house are pointing to a particular agent.

After this tentative assignment x, there is no good cycle among the subagents and the whole graph is a strongly connected absorbing set. Hence, x is in fact the final assignment.

When dealing with trades, a concern may be that even if the algorithm terminates, it may converge to an efficient solution very slowly. A natural approach is to 'discretize' a fractional assignment problem by breaking each house into a small enough mini-house. However, this approach can lead to an exponential blowup. Next, we show that FTTC terminates in time polynomial in m and n.

Theorem 1. *FTTC terminates in time $O(m^3n^4)$.*

5 Axiomatic Properties of FTTC

We examine the axiomatic properties satisfied by FTTC. Firstly, we observe that during each trade in FTTC the agents' allocation gets an SD-improvement because for each agent in a trade, some units of a house are replaced by an equal units of a house that is at least as preferred. Hence SD-individually rationality is easily satisfied.

Theorem 2. *FTTC is SD-individually rational.*

Theorem 3. *FTTC is SD-efficient.*

In what follows, we will prove that FTTC also SD-core stable. In order to make the argument, we present a connection with an *associated cloned housing market*. For a given fractional housing market $(N, H, \succsim, e)$ and a small enough $\epsilon > 0$, the *associated cloned housing market* with respect to ϵ is a housing market $(N, H', \succsim', e')$ where N' consists of subagents of agents in N, each agent in N' is endowed with exactly ϵ units of a single house, and the sum of the endowments of the subagents of an agent is equal to the endowment of the agent. For a cloned market $(N, H', \succsim', e')$ and its assignment x', its corresponding assignment for the original market $(N, H, \succsim, e)$ is assignment x such that $x(i) = \sum_{j \text{ subagent of } i} x'(j)$.

Lemma 1. *If the assignment x' of the cloned housing market is core stable, the corresponding assignment x in the original fractional housing market is SD-core stable.*

We can then use Lemma 1 to prove that FTTC is SD-core stable.

Theorem 4. *FTTC is SD-core stable.*

The desirable aspect of FTTC is that under discrete and single-unit endowments, it additionally satisfies other desirable properties such as strategyproofness and strict core stability.

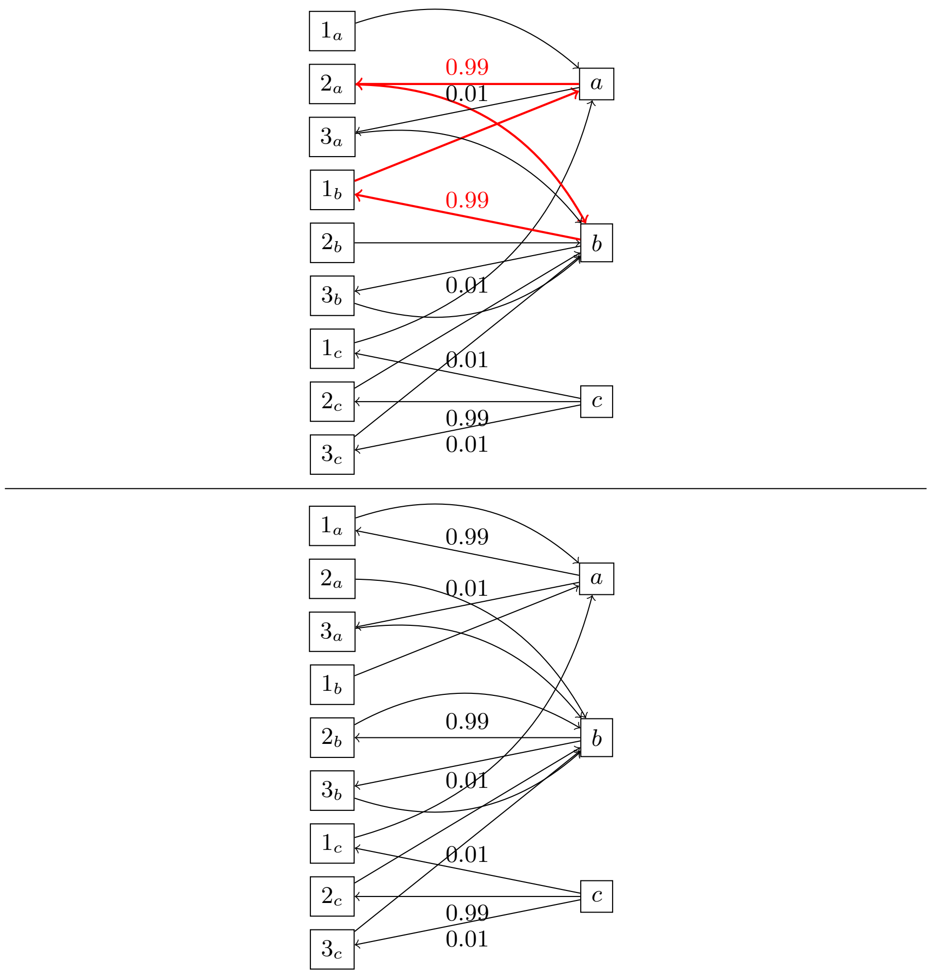

Fig. 1. Illustration of the graph during the running of FTTC. The graph in the first step admits a good cycle that is highlighted in bold red. Since the graph in the second step admits no good cycle and since the whole graph is an absorbing set, hence the assignment corresponding to the graph is finalized.

Theorem 5. *For the classic housing markets with indifferences (with discrete and single-unit endowments), FTTC is core stable, Pareto optimal, and strict core stable (if the strict core is non-empty), and strategyproof.*

Proof. Whenever $e(i)(h) \in \{0,1\}$ and $\sum_{h\in H} e(i)(o) = 1$ for each $i \in N$, we know that whenever a good cycle is implemented, $\alpha = 1$ which means that a complete house is exchanged since each agent owns a single complete house. Hence is easily seen that FTTC is a GATTC mechanism as defined in [6]. It follows there for that for discrete and single unit endowments, it is core stable, Pareto optimal and strict core stable (if the strict core is non-empty). As for strategyproofness, note that good cycles are implemented exactly as by the algorithm of Plaxton [18]. Thus strategyproofness is also ensured. Note that for the classic housing market with indifferences, each house is possessed by at most one agent at any point. Therefore, the tie-breaking over the set of agents is redundant. □

Corollary 1. *For the classic housing markets without indifferences, FTTC is equivalent to the TTC mechanism.*

Proof. The statement follows from the fact that GATTC coincides with TTC for strict preferences and discrete endowments. □

Note that as long as each agent is endowed one house and each house has the same number of units in the market, then FTTC runs in the same way as if there was one unit of each house in the market. Hence, FTTC is strategyproof as long as each agent is endowed one house and each house has the same number of units in the market.

As a result of Theorem 4, and Corollary 1, FTTC constitutes a generalization of TTC that still satisfies core stability in the fractional setting. FTTC also generalizes the ATTC mechanism [11] for housing markets with strict preferences, discrete but multi-unit endowments.

Theorem 6. *For the housing markets with strict preferences, discrete but multi-unit endowments, FTTC is equivalent to the ATTC mechanism.*

Proof. Note that ATTC is equivalent to first forming a cloned housing market in which each subagent of an agent owns exactly one house and then running TTC over the cloned housing market. Equivalently, when endowments are discrete and preferences are strict, then FTTC runs in exactly the same manner. □

An interesting corollary of Theorem 6 is that manipulating FTTC with respect to SD is NP-hard.

Corollary 2. *Manipulating FTTC with respect to SD is NP-hard.*

Proof. Fujita et al. [11] showed that manipulating ATTC is NP-hard for agents with lexicographic preferences. The same proof can also be used to show that manipulating ATTC is NP-hard for agents with strict preferences over houses that are extended to fractional allocations via stochastic dominance. Since FTTC coincides with ATTC for strict preferences and discrete endowments, it follows that manipulating FTTC with respect to both DL (lexicographic preferences) and SD is NP-hard. □

6 Impossibilities

We showed that FTTC is SD-core stable. It is also known that the CC rule satisfies NJE. This raises the question whether FTTC which is SD-core stable can additionally satisfy no justified envy or whether CC satisfies SD-core stability. Next we show that no justified envy and SD-core stability are in fact incompatible.

Theorem 7. *There exists no mechanism that satisfies both SD-core stability and no justified envy even for single unit allocations and endowments.*

Proof. Consider the following housing market $(N, H, \succsim, e)$ where $N = \{1, 2, 3\}$, $H = \{a, b, c\}$, The endowment assignment can be represented as follows:

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$\begin{aligned} 1: &\quad c, b, a \\ 2: &\quad a, b, c \\ 3: &\quad a, b, c \end{aligned}$$

The *only* core stable assignment is one which both agent 1 and 2 gets one unit of their most preferred house: $x = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

But in assignment x, agent 3 has justified envy towards agent 2 because $x(2) \succ_3^{SD} x(3)$ and $x(3) \succsim_2 e(2)$. □

Corollary 3. *FTTC does not satisfy no justified envy.*

Corollary 4. *CC is not SD-core stable*

One may wonder whether FTTC is weak SD-strategyproof. We show that this is impossible because there cannot exist an SD-efficient, SD-individually rational and weak SD-strategyproof mechanism. The theorem below answers a question raised by Athanassoglou and Sethuraman [2].

Theorem 8. *There does not exist a weak SD-strategyproof, SD-efficient and SD-individually rational fractional housing market mechanism even for single unit allocations and endowments.*

Corollary 5. *There does not exist a weak SD-strategyproof, SD-efficient and SD-core stable fractional housing market mechanism.*

We also get as corollaries previous impossibility results in the literature:

Corollary 6 (Theorem 4, Yilmaz [29]). *There does not exist an SD-IR, SD-efficient, weak SD-strategyproof, and weak SD-envy-free fractional housing market mechanism.*

Corollary 7 (Theorem 2, Athanassoglou and Sethuraman [2]). *There does not exist an SD-IR, SD-efficient, weak justified envy-free and weak SD-strategyproof fractional housing market mechanism.*

Corollary 8 (Theorem 3, Athanassoglou and Sethuraman [2]). *There does not exist an SD-IR, SD-efficient, and SD-strategyproof fractional housing market mechanism.*

We remark that the three properties used in Theorem 8 are independent from each other.SD-efficiency and weak SD-strategypoofness can be simultaneously satisfied by the *multi-unit eating probabilistic serial mechanism* [5, 13] if preferences are strict. SD-individual rationality and weak SD-strategyproofness (even SD-strategyproofness) are satisfied by the mechanism that returns the endowment. SD-individual rationality (even SD-core) and SD-efficiency are satisfied by FTTC.

One may hope that the strict core is non-empty for dichotomous preferences. Unfortunately, this is not the case even if fractional allocations are allowed. The following example shows that for the housing markets with dichotomous preferences, the strict core can be empty for both fractional allocations and discrete allocations.

Example 3. Consider the following housing market $(N', H, \succsim, e)$ where $N = \{1, 2, 3, 4, 5\}$, $H = \{h_1, h_2, h_3, h_4, h_5\}$, $e(i)(h_i) = 1$ and zero otherwise. The set of approved houses for each agent are as follows: $\max_{\succsim_1}(H) = \{h_1, h_2, h_4\}$; $\max_{\succsim_2}(H) = \{h_3\}$, $\max_{\succsim_3}(H) = \{h_1\}$; $\max_{\succsim_4}(H) = \{h_5\}$, $\max_{\succsim_5}(H) = \{h_1\}$. Then it is clear that agent 1 can toggle between enabling cooperation between h_2 and h_3 or h_4 and h_5.

7 Discussion and Conclusions

	CC	FTTC
SD-individual rationality	+	+
SD-core stability	−	+
weak SD-strategyproof	−	−
no justified envy	+	−
polynomial-time	+	+

Table 2. Properties satisfied by mechanisms for allocation of houses under fractional endowments.

In this paper, we proposed a general mechanism for housing markets with fractional endowments. Just as TTC is SD-efficient and SD-core stable for the discrete single-unit domain, FTTC satisfies the same properties on the more general domain. Since FTTC is SD-core stable and the Controlled Consuming

(CC) algorithm of Athanassoglou and Sethuraman [2] is not, FTTC has at least one advantage over a previously introduced mechanism for fractional housing markets. However it cannot additionally satisfy no justified envy because the property is incompatible with SD-core stability. Whereas CC coincides with the PS algorithm of Bogomolnaia and Moulin [10] under equal endowments, FTTC does not. On the other hand, FTTC coincides with the TTC under full endowments but CC does not. Table 2 summarizes the properties satisfied by CC and FTTC.

Note that the FTTC can be easily extended to the case where some houses are not endowments: simply make each agent have endowment $1/n$ of each public house or give the houses to the agent(s) with the highest priority. In this way, FTTC can also be used as an interesting way to allocate houses when none of the agents are endowed: initially give $1/n$ of each house to each agent and then run FTTC on the created housing market. The outcome is SD-proportional as defined by Aziz et al. [7].

FTTC can also be adapted to cater for agents expressing some houses as unacceptable: agents are never made to point to unacceptable houses. We also note that when agents have discrete endowments, then all our positive results with respect to SD carry over to positive results when agents have preferences over sets of houses via the responsive set extension [4, 8, 20].

We showed that FTTC is not weak SD-strategyproof. It will be interesting to identify restricted preferences or domains under which FTTC is strategyproof. In particular, we conjecture that FTTC is strategyproof for 0-1 utilities.

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Proof of Theorem 1

Proof. In each iteration of the algorithm, absorbing sets are computed which takes linear time on the graph $O(nm)$. Furthermore, it can be checked in linear time whether a given absorbing set is non-good or not by checking whether a subagent points to a strictly more preferred house or not. If there exists at least one non-good absorbing set at least some vertices are deleted are from the graph (in Step 9) which means that there can be at most $O(nm)$ iterations.

In case there is no non-good absorbing set, we show that implementing a cycle cannot happen indefinitely and that after $O(m^2n^2)$ cycles are implemented, there may not remain any good cycle or at least a non-good absorbing set appears. If there does not exist a non-good absorbing set, there is at least one good cycle in the graph. All vertices other than attractors are made to point to vertices with the shortest distance to some attractor. This means that in each good cycle C, there is at least one attractor. If each time an attractor loses all the weight α of the house he owns, there can be at most $O(mn)$ cycles until there exists no good cycle. Now let us assume that we encounter a series of cycles in which no attractor completely loses his house. We show there can be only a bounded number of such cycles.

If cycles are repeated and no attractor loses all of his house, this means that no attractor can become a non-attractor. A non-attractor subagent who loses his house does not become an attractor by definition since an attractor i_h owns non-zero fraction of h. It may be the case be that a non-attractor subagent i_h who did not have any units of h may gain a fraction of house h due to his fellow subagent getting a most preferred house h. But this means that i_h is not an attractor since it is already partially owns a maximally preferred house in the graph. Thus no non-attractor becomes an attractor and no attractor becomes a non-attractor in these series of cycles. Note that the weights of some edges change, and the distance values of the vertices in the graph can only increase because edges towards attractors are deleted. Let the cycle include the sequence $i'_{h'}, h, i_h, h^*, j_{h^*}$ where i is the agent with the shortest path to an attractor who loses his ownership of h since h now points to a subagent of i' instead of i. Since h^* was not completely taken away from j_{h^*}, it still points to j_{h^*} that still points to the same most preferred house with shortest distance to an attractor that he was pointing to before implementing the cycle. Hence the distance value of h^* does not change. After the cycle is implemented, there is no longer a shortest path of $i'_{h'}$ to an attractor that includes h and i_h. One way, subagents of i' may have a path $i'_{h'}, h, i_h, \ldots,$ towards an attractor is if i_h again gets ownership of some fraction of h from some other cycle. But note that i_h as well as all other subagents of i_h were uniquely being made to point to h^* and they will keep pointing to h^* until h^* is no more in the graph or the distance value of h^* changes. Hence these kinds of cycles can occur at most mn times before there are no paths from i' to an attractor. Therefore, there can be at most $O(m^2n^2)$ implementations of such cycles before there is no path from a non-attractor to an attractor whereafter every good cycle only involves attractors. This means that we have now encountered a non-good absorbing set and also that each good cycle involves an attractor losing all of his partial ownership.

Therefore after at most $O(m^2n^3)$ cycles at least one non-good absorbing set is encountered and there can be at most $O(mn)$ absorbing sets that can be deleted which means that the algorithm runs in time $O(m^3n^4)$. □

A Proof of Theorem 3

Given random assignments x and y, $x(i) \succsim_i^{DL} y(i)$ i.e., an agent i *DL (downward lexicographic) prefers* allocation $x(i)$ to $y(i)$ if $x(i) \neq y(i)$ and for the most preferred house h such that $x(i)(h) \neq y(i)(h)$, we have that $x(i)(h) > y(i)(h)$. Next, we prove that FTTC returns a DL-efficient assignment which implies SD-efficiency. In order to prove it, we use the following lemma.

Lemma 2. *Assume that $y(i) \succsim_i^{DL} x(i)$ for all $i \in N$. Also for each $S \in \{E_i^1, \dots, E_i^{k_i}\}$, $\sum_{h \in S} y(i)(h) \leq \sum_{h \in S} x(i)(h)$. Then $y(i) \sim_i^{DL} x(i)$ for all $i \in N$.*

Theorem 9. *FTTC is DL-efficient.*

Proof. Let the assignment returned by FTTC be x. Assume for contradiction that x is not DL-efficient. Then, there exists an assignment y such that $y(i) \succsim_i^{DL} x(i)$ for all $i \in N$ and $y(i) \succ_i^{DL} x(i)$ for some $i \in N$. Let the absorbing sets with no good cycle during the running of FTTC be $S_1, \dots S_o$. Let the most preferred houses in S_t for agent be $\max_{\succsim_i}(S_t)$. For any set of equally preferred house H', let $E_i(H') = \{h \in H : h \sim_i h' \in H'\}$. Let $J(t)$ be the following statement:

> $J(t)$: for the t-th non-good absorbing set encountered during the running of FTTC,
> $$y(i)(E_i(\max_{\succsim_i}(S_t))) \leq x(i)(E_i(\max_{\succsim_i}(S_t))).$$

We prove by induction that $J(1), \dots, J(o)$ hold.

Base Case $J(1)$ is the following statement: for the 1-st absorbing set, $y(i)(\max_{\succsim_i}(S_1)) \leq x(i)(\max_{\succsim_i}(S_1))$ for all i who have a subagent in S_1. Note that each i who has subagent in S_1 points to all the houses in the first equivalence class. Moreover all such houses are in S_1 or else S_1 would not be an absorbing set. Since $y(i) \succsim_i^{DL} x(i)$ for all $i \in N$, it follows that for agents represented in S_1, each agent gets at least as much units of his first equivalence class in y as in x. Since all the houses in S_1 are allocated to agents represented in S_1 and since each of the agents only get units of their most preferred houses, if one of the represented agents gets strictly more units of his first equivalence class, then it means that at least one represented agent in S_1 gets less units of his first equivalence class. But this is not possible since $y(i) \succsim_i^{DL} x(i)$ for all $i \in N$.

Induction Assume $J(1)$ to $J(\ell)$ holds. When FTTC consider $S_{\ell+1}$, all the houses in S_1 to S_ℓ are fixed and each of the represented agents in those sets get exactly the same units of their most preferred equivalence classes as in x. We now focus on $S_{\ell+1}$. Note that each i who has a subagent in $S_{\ell+1}$ points to all the most preferred houses that have not been deleted (and whose allocations have not been finalized). Moreover all such houses are in $S_{\ell+1}$ or else $S_{\ell+1}$ would not be an absorbing set. Since $J(1)$ to $J(\ell)$ holds and $y(i) \succsim_i^{DL} x(i)$ for all $i \in N$, it follows that for agents represented in $S_{\ell+1}$, each agent gets at least as much units of his current equivalence class in y as in x. Since all the houses in $S_{\ell+1}$ are allocated to agents represented in $S_{\ell+1}$ and since each of the agents only get units of their most preferred remaining houses, if one of the represented agents strictly more units of his current maximal equivalence class, then it means that at least one represented agent in S_1 gets less units of his current maximal equivalence class. But this is not possible since $y(i) \succsim_i^{DL} x(i)$ for all $i \in N$.

Hence, we have shown that for each $S \in \{E_i^1, \ldots, E_i^{k_i}\}$, $\sum_{h\in S} y(i)(h) \leq \sum_{h\in S} x(i)(h)$. Since we assumed that $y(i) \succsim_i^{DL} x(i)$ for all $i \in N$, by Lemma 2, we get that $y(i) \sim_i^{DL} x(i)$ for all $i \in N$. This is a contradiction because we assumed that $y(i) \succsim_i^{DL} x(i)$ for all $i \in N$ and $y(i) \succ_i^{DL} x(i)$ for some $i \in N$. □

Proof of Theorem 4

Proof. Let ϵ be greatest common divisor of fractional allocations of the houses to agents in allocation x. We first view FTTC as a discrete algorithm in which each agent is represented by sufficient number of clones and each clone owns exactly one mini-house where the sizes of all mini-houses is the same. Thus there may be many clones of agent i corresponding to each subagent of agent i. During the running of FTTC, when a subagent i_h and h a most preferred house in the graph are removed from a non-good absorbing set with $w(h, i_h) = \lambda$, we will view this as λ/ϵ clones of i being removed from the graph. At that point the allocation of the clones is fixed in the discrete view of the algorithm and the allocation of h for the subagent i_h and hence of i is fixed in the actual view of the FTTC algorithm. Then, when a clone's allocation is fixed in FTTC, then it cannot be part of any weakly blocking coalition since it gets a most preferred mini-house in the graph. So for the base case, the subagents whose allocation is fixed in the first non-good absorbing set cannot be part of a weakly blocking coalition. Now by the same argument, given that subagents whose allocations have already been set will not be part of the weakly blocking coalition, then the next subset of subagents whose allocation gets fixed will not be part of a weakly blocking coalition. By induction, no agent will be a member of a weakly blocking coalition. Thus we have proved that the assignment x' for the cloned housing market is core stable.

Since the cloned assignment x' is core stable, it follows from Lemma 1 that the assignment x returned by FTTC is SD-core stable. □

Proof of Theorem 8

We first prove Lemma 1.

Proof. We show that if x is not core stable, then x' is not core stable. If there exists a coalition $S \subset N$ weakly blocking x, then each of the agents in $i \in S$ gets an SD-better outcome $y(i)$ by simply trading among the agents in S. For each agent in $i \in S$, let us compare $y(i)$ with $x(i)$. We know that $y(i) \succ_i^{SD} x(i) \succsim_i^{SD} e(i)$. For each $i \in S$, consider the most preferred equivalence class E such that $y(i)(E) > x(i)(E)$. The extra probability weight of E that i gets is because of the endowment of other agents in S. Thus for each $i \in S$, there exists some $h \in E$ such that $y(i)(h) > x(i)(h) \geq e(i)(h)$. Let each agent $i \in S$ point to another agent j such that $e(j)(h) > 0$. Since each agent $i \in S$ got a strict SD-improvement in $y(i)$ over $x(i)$, this means that each agent points to at least some other agent. Hence there exists a cycle C among a subset $S' \subset S$ in which in which each agent i gives up ϵ units of a house less preferred than h and owned by j in $e(j)$ and gets ϵ units of h. Now consider the set S' consisting of one clone each of agents in S where each clone in S' can get strictly more preferred mini-house in cloned assignment y' where the mini-house was originally owned by a clone of an agent $j \in S$. Hence the cloned agents in S' form a coalition weakly blocking assignment x'. This means that the assignment x' is not core stable. □

Next, we present the proof of Theorem 8.

Proof. Consider the housing market $(N, H, \succsim, e)$ where $N = \{1, 2, 3, 4, 5\}$, $H = \{h_1, h_2, h_3, h_4, h_5\}$, the preference profile $\succsim$ is

$$\begin{aligned}
\succsim_1 &: \quad h_3, h_1, h_2, h_4, h_5\\
\succsim_2 &: \quad h_5, h_1, h_2, h_3, h_4\\
\succsim_3 &: \quad h_1, h_4, h_2, h_3, h_5\\
\succsim_4 &: \quad h_2, h_4, h_1, h_3, h_5\\
\succsim_5 &: \quad h_5, h_3, h_1, h_2, h_4
\end{aligned}$$

and

$$e = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0\\ 0 & 0 & 1/2 & 0 & 1/2\\ 1/2 & 0 & 0 & 1/2 & 0\\ 0 & 1/2 & 0 & 1/2 & 0\\ 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

Assume that agents 3, 4 and 5 do not misreport. We show that in this case they get exactly their endowment under some weak constraints on the preferences of agents 1 and 2.

In any SD-individually rational assignment, agent 3 gets at least $1/2$ of h_1 and remaining weight of h_4 to sum up to one unit. Similarly, agent 4 gets at least

1/2 of h_2 and remaining weight of h_4 to sum up to one unit. Agents 3 and 4 can get more of their most preferred house if they can exchange their second most preferred house h_4 for more of their most preferred house. However, as long as h_4 is less preferred than h_1, h_2, h_3 for agent 1, h_4 is the least preferred for agent 2 and agent 5, then no agent will be willing to get more of h_4 and give either h_1 or h_2 to agents 3 and 4.

Hence agent 3 and 4 get exactly their endowment as long as agents $3, 4, 5$ report truthfully, agent 2 reports h_4 as his least preferred house, and agent 1 expresses h_4 as a house less preferred than h_1, h_2, h_3.

Due to the SD-individually rationality requirement, agent 5 has to get at least 1/2 of h_5 and the remaining of h_3 to get a total of one unit. Agent 5 cannot get more than 1/2 of h_5 because agent 2 must get 1/2 of h_5 as well. We conclude that agent 5 gets exactly 1/2 of h_5 as long as agent 2 expressed h_5 as his most preferred house. Due to SD-IR, agent 5 must get exactly his endowment as long as agent 2 expressed h_5 as his most preferred house.

Thus we have established that agents $3, 4, 5$ get exactly their endowment as long as the following conditions hold:

- agents $3, 4, 5$ report truthfully
- agent 2 reports h_4 as his least preferred house
- agent 2 reports h_5 as his most preferred house
- agent 1 expresses h_4 as a house less preferred than h_1, h_2, h_3.

From now on, we will consider preference profile in which the conditions above are met so that by SD-IR, we get that agents 3, 4, 5 get exactly their endowments.

Assuming that agents $3, 4, 5$ get the same allocation as their endowment, agent 1 must get 1/2 of h_3 in any SD-efficient assignment. Thus the only SD-individually rational and SD-efficient assignments for profiles satisfying the conditions above:

$$x = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix},$$

$$y = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix},$$

and

$$z = \begin{pmatrix} \lambda & 1/2-\lambda & 1/2 & 0 & 0 \\ 1/2-\lambda & \lambda & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

for $0 < \lambda < 1/2$.

If the outcome is assignment x, then agent 2 can report $\succsim'_2$:

$$\begin{aligned} \succsim_1&: \quad h_3, h_1, h_2, \ldots \\ \succsim'_2&: \quad h_5, h_1, h_3, h_2, h_4 \end{aligned}$$

The only SD-individually rational and SD-efficient outcome of $(\succsim_1, \succsim'_2, \succsim_3, \succsim_4, \succsim_5)$ is assignment y which is an SD-improvement for agent 2 over the truthful outcome x.

If the outcome is assignment y, then agent 1 can report $\succsim'_1$:

$$\begin{aligned} \succsim'_1&: \quad h_1, h_3, h_2, \ldots \\ \succsim_2&: \quad h_5, h_1, h_2, h_3, h_4 \end{aligned}$$

The only SD-individually rational and SD-efficient outcome of $(\succsim'_1, \succsim_2, \succsim_3, \succsim_4, \succsim_5)$ is assignment x which is an SD-improvement for agent 1 over the truthful outcome y.

If the outcome is of type assignment z, then agent 1 can report $\succsim'_1$:

$$\begin{aligned} \succsim'_1&: \quad h_1, h_3, h_2, \ldots \\ \succsim_2&: \quad h_5, h_1, h_2, h_3, h_4 \end{aligned}$$

The only SD-individually rational and SD-efficient outcome of $(\succsim'_1, \succsim_2, \succsim_3, \succsim_4, \succsim_5)$ is assignment x which is an SD-improvement for agent 1 over the truthful outcome z. □