Cooperation of groups - an optimal transport approach

Xinyang Wang

Yale University, Department of Economics

Abstract. In this paper, we study the assignment problem with a bound on group size. In particular, we link the assignment problem with finite group size bound and positive group size bound to multi- and continuum-marginal optimal transport (Monge-Kantorovich) problem and show the existence of a stable assignment. Without demanding the compactness and non-atomicness assumptions on agent space, we provide a clean model as well as tools to unify a number of classical results including the existence of fractional core, f-core and epsilon-sized core, and solve the problem of continuum extension of Gale and Shapley's work in [11] proposed by Kaneko and Wooders in [16].

1 Introduction

In cooperative game (or TU-game) theory, it is a central question that, given the surplus each subset of agents could generate, if there is a stable way to divide the surplus. The stability here means no subset of agents wish to deviate and generate a higher surplus. This notion of stability, defined as core, was firstly proposed by Edgeworth [10] and then related to game theoretic setting by Gillies [13].

Gale and Shapley [11] observed group size might be bounded exogenously (e.g. in marriage market) and initiated the study of matching problem and roommate problem in a non-transferable utility model. Later, Scarf [21], Shapley and Shubik [24] studied its transferable utility counterpart. Shapley and Shubik named this classic problem by "Assignment Problem". The work following this trajectory has been extensive, including generalizations to large agent space in [14], [8], [5], [7]. These works focused on the case when the group size is bounded by two. On the contrary, when there is no restrictions on the group size, Bondareva [2] and Shapley [23] established existence results for a discrete agent space in a transferable utility environment and Scarf [21] extended the results to a non-transferable utility environment.

In this paper, we study something in the middle - assignment problem with an arbitrarily given bound on group size. In particular, this size bound could be a finite bound on the number of group members or a positive bound on the measure of group members. The problem is stated as following: given a space of agents (not necessarily finite or compact), each subset of agents of bounded size, namely a group, is able to generate a surplus which they could divide in any way within the group. An assignment, generalized from a matching, is a plan instructing groups to form. We say an assignment is stable if there is a way to divide payoffs among agents such that this division is both feasible and that no group has incentive to deviate by generating a higher surplus.

We show for any given bound on group size, whenever the surplus function is upper semi-continuous and bounded in a very mild sense, a stable assignment always exists. In particular, we do not assume any relationship between surplus functions on different sizes of groups. Consequently, our model includes the case that a third person could strictly decrease the surplus of a two-person group, and the resulting stable assignment forms agents into groups of different sizes. On a technical level, contrary to [15], [16], [14], we do not assume agent space to be compact or endowed with a non-atomic measure, thus our result could be degenerated to a finite type space.

The relationship between assignment problem and linear programming problem was firstly introduced by Bondareva [2], Shapley [23], Scarf [21] and Shapley and Shubik [24] in a discrete setting and later extended by Gretsky, Ostroy and Zame [14] to a continuum setting. Both work focused on the case that group size bound is exactly two. In addition, Kaneko and Wooders in [16] and [15] studied the case when the group size is bounded from above by a finite number and claimed the stable notion exists in an approximate sense by exploring the compactness of agent space. On a separate direction, Schmeilder [22] showed that core is an invariant concept under positive group size bound. However, his elegant proof relies on Lyapunov theorem, thus only working in an exchange economy with finite dimensional commodities spaces but failing (as we shall see in the paper) in the more general cooperative game setting this paper studies.

Recently, Chiaporri, McCann, Nesheim and Pass in [8] and [7] analyzed explicitly the relationship between matching problem and optimal transport problem. In this paper, we explore in their direction. In particular, we link the problem with finite bound to a multi-marginal transport problem and the problem with positive bound to a "continuum"-marginal transport problem. While the classical 2-maringal problem is studied extensively as shown in Villani's book [25], there are many gaps in the general theories in multi- and "continuum"- marginal optimal transport problem. Some partial results could be located in [18], [19], [20].

Similar to [14], we prove our theorems by establishing three propositions:

1. the existence of welfare maximizing assignment
2. the duality between welfare maximization and constrained utility minimization
3. the equivalence of stable assignments and constrained utility minimization

The first proposition corresponds to solution existence of Kantorovich tranport problem. The second proposition corresponds to Kantorovich-Koopmans duality. And the third proposition relates the concept of stability to the solution of minimization problem.

In conclusion, by linking the assignment problem with finite and positive bounds on group sizes to optimal transport problem, we unify and generalize various previous results, including [21], [24], [14], [15], [8], on core existence.

The paper is organized as follows. Section 2 introduces a model with a finite size bound. Section 3 establishes and proves the existence result for finite size bound case. Section 4 introduces a model with a positive size bound. Section 5 establishes and proves the existence result for positive size bound case. Section 6 summarizes the results and discusses possible future work.

1.1 Notations

We list notations we used in this subsection:

Definition 1. *For a complete separable metric space I, we define*

- $\mathcal{B}(I)$*: Borel sigma-algebra of I*
- $\mathcal{M}_+(I)$*: the space of non-negative Borel measures on I*
- $\mathbb{P}(I)$*: the space of Borel probability measures on I*
- $C_b(I)$*: Bounded continous functions on I.*
- $O(n)$*: permutations of n elements. We identify them by maps from* I^n *to* I^n *that permutes the coordinates.*
- $\pi_n : I^N \to I$*: the projection operator on n-th coordinate, where* $n \leq N$*.*

Definition 2. *For* $\mu_1, \mu_2 \in \mathcal{M}_+(I)$*, we say* $\mu_1 \leq \mu_2$ *if for any* $A \in \mathcal{B}(I)$*, we have* $\mu_1(A) \leq \mu_2(A)$*.*

Definition 3. *For a complete separable metric space I and a* $\mu \in M_+(I)$*,* $\mu \neq 0$*, define* $L^1(I, \mu)$ *be the space of integrable functions, i.e.* $\int_I |f| d\mu < +\infty$ *for any* $f \in L^1(I, \mu)$*. When there is no confusion on underling measure, we write the space as* $L^1(I)$*.*

Definition 4. *If* μ *is a Borel measure on X, T is a Borel map from X to Y, then we denote the image measure (or push-forward measure) of* μ *by* $T_\#$ *such that for any* $A \in \mathcal{B}(Y)$*,*

$$T_\#\mu(A) = \mu(T^{-1}(A))$$

In this paper, we will mainly use this definition in the following two ways: Firstly, for a measurable set $A \subset I$, and a measure γ_n on I^n,

$$(\pi_1)_\#\gamma_n(A) = \gamma_n(A, I, ..., I)$$

Secondly, for a permutation $\sigma_n \in O(n)$, and a measure γ_n on I^n, $A_1, ..., A_n$ are measurable sets on I^n, we have

$$(\sigma_n)_\#\gamma_n(A_1, ..., A_n) = \gamma_n(A_{\sigma(1)}, A_{\sigma(2)}, ..., A_{\sigma(n)})$$

We will discuss in more details when we use any of them.

2 Model for groups of finite size

2.1 Environment

We study a cooperative game $((I, \mu), v, N', N)$ defined as follows. Let a Polish space [1] I be the type space of agents and $\mu \in \mathbb{P}(I)$ be the distribution of agents' types.

In this section, we study the case when the size bounds are finite. We use $N \in \mathbb{N}$ to denote the upper bound on group size and use $N' \leq N$ where $N' \in \mathbb{N}$ to denote the lower bound on group size.

For $N' \leq n \leq N$, the set of groups of size n is identified by I^n, where I^n is the n-folds Cartesian product of agent space I. Each element $C = (i_1, ..., i_n) \in I^n$ corresponds to a group containing n agents of types $i_1, ..., i_n$. Note, as we treat I as the type space, we allow a group contains multiple agents of the same type. Moreover. the ordering does not matter when describing a group. That is, for any permutation $\sigma \in O(n)$, $(i_1, ..., i_n)$ and $(i_{\sigma(1)}, ..., i_{\sigma(n)})$ refer to the same group. The metric on I^n is the product metric induced from I. The set of groups is denoted by $\cup_{n=N'}^{N} I^n$.

Next, we specify the surplus each group could generate by using a function. Mathematically, surplus function $v : \cup_{n=N'}^{N} I^n \to \mathbb{R}$ is a real-valued function defined on the set of groups. For clarity, we express v by a tuple $(v_{N'}, ..., v_N)$, where v_n is a real-valued function on groups of size n, I^n, for $N' \leq n \leq N$. We will need the following assumptions on the surplus function v:

(A1) v is component-wisely upper-semi continuous: For any $N' \leq n \leq N$, v_n is upper semi-continuous. i.e.

$$\forall C_k, C \in I^n, C_k \to C, \text{then}, \limsup_{k\to\infty} v_n(C_k) \leq v_n(C)$$

(A2) v is symmetric: For any $N' \leq n \leq N$, and any permutation $\sigma \in O(n)$, $(i_1, ..., i_n) \in I^n$,

$$v_n(i_1, ..., i_n) = v_n(i_{\sigma(1)}, ..., i_{\sigma(n)})$$

(A3) v is bounded from above: For any $N' \leq n \leq N$, there is lower semi-continuous functions $a_n \in L^1(\mu)$ such that,

$$v_n(i_1, ..., i_n) \leq \sum_{j=1}^{n} a_n(i_j)$$

The first assumption states the regularity requirement. Weaker than the continuity requirement, this assumption allows us to take the case that some groups have a discontinuously large surplus into account. The second assumption states the consistency requirement: as $(i_1, ..., i_n)$ and

[1] A Polish space is a separable completely metrizable space. Examples are, in the general equilibrium setting, an agent's type is denoted by $(x, u) \in B \times C_b(B)$, where B is some bounded set determined by the total resources. When the underlining measure on type distribution is non-atomic, the measure space is isomorphic to [0,1] endowed with Lebesgue measure. Here, we do not impose the non-atomic requirement.

$(i_{\sigma(1)}, ..., i_{\sigma(n)})$ denote the same set of agents, they have the same surplus. The third assumption is a technical assumption guaranteeing the integrability of the surplus function v. This assumption is not restrictive as it is sufficient to assume v_n is bounded from above by a constant.
Note we assume no relationship between surplus functions of different coalition sizes. i.e. there is no relation between v_n and v_m when $n \neq m$. Indeed, for any group, the advent of one more member may discontinuously increase or decrease the surplus of this group. Thus, we do not rule out externality here, and study a broader class of models than super-additive games.
Indeed, the surplus of a group could have a jump (upward or downward) when one agent outside the group joins or one agent in the group leaves.

2.2 Group assignment

We use assignment to describe the way agents are grouped. Note, similar to Scarf's work, we propose this notion in a random manner.
It is worth noting that assignment is defined to be a random measure in this paper. In contrast to their notions about pure assignment in [11], [5] in which all agents of the same type must be assigned in the same way, our definition, similar to that in [21], [24],[16], [14], [8], allows assigning different agents of the same type in different ways. For instance, as shown in Figure 1, there are three types of agents and each type has a mass 1/3. When the group size is bounded above by 2, a pure assignment must leave one person alone if a group of size 2 if formed. On the contrary, a random assignment can assign half of type 1 to a group with type 2 and half of type 1 to a group with type 3.

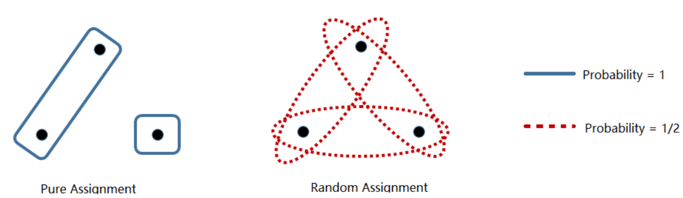

Fig. 1. Pure assignment and random assignment

Formally, an assignment is a tuple $\gamma = (\gamma_{N'}, ..., \gamma_N)$ such that

1. for all $N' \leq n \leq N$, γ_n is a positive measure on I^n
2. $\sum_{n=N'}^{N} (\pi_1)_{\#}\gamma_n = \mu$
3. for all $N' \leq n \leq N$, $\sigma_n \in O(n)$, $(\sigma_n)_{\#}\gamma_n = \gamma_n$

Moreover, we use Γ_{sym} to be the set containing all assignments γ.
For both expositional and technical reasons, we separate the study of the special case where $N' = N$. In this case, an assignment γ is defined to be a probability measure on I^N satisfying the latter two conditions:

2′ $(\pi_1)_{\#}\gamma_n = \mu$

3′ For all $\sigma_n \in O(n)$, $(\sigma_n)_{\#}\gamma = \gamma$

Intuitively, for any group $C = (i_1, ..., I_n) \in I^n$, $\gamma(C)$ is $1/n!$ times the probability the set of agents $\{i_1, ..., i_n\}$ forming a group according to γ. The coefficient $1/n!$ appears as the set of agents $\{i_1, ..., i_n\}$ corresponds to $n!$ elements in I^n. Condition 2′ states market clearing condition that all agents are assigned into some groups. To describe it, we note for any subset of types $A \subset I$, $\gamma_n(A, I, ..., I)$ denotes the probability agents of types belonging to A are assigned into some group. And this number should equal to the measure of this subset A. That is, $\gamma_n(A, I, ..., I) = \mu(A)$. Condition 3′ states the symmetry requirement that reordering of members in a group will not change the probability a group is formed.

More generally when $N' \neq N$, any agent should be assigned by γ into a group of some size $n \in [N', N]$. Therefore, we can view each γ_n as a grouping instruction for the subset of agents who will be assigned into groups of size n. Intuitively $\gamma_n(i_1, ..., i_n)$ is $1/n!$ times the probability the set of agents $\{i_1, ..., i_n\}$ forming a group according to γ. However, as the subset of agents assigned to groups of the given size may not have the full measure, condition 1 requires γ_n to be a positive measure, rather than a probability measure, on I^n. Condition 2 and 3 are similar to the previous special case: Condition 2, which is

$$\forall A \in \mathcal{B}(I), \sum_{n=1}^{N} \gamma_n(A, I, I, ..., I) = \mu(A)$$

is the market clearing condition which states the mass of agents of types belonging to A assigned by assignment γ is equal to the mass of agents of types belonging to A. Condition 3, which is

$$\forall \sigma_n \in O(n), \forall A_1, ..., A_n \in \mathcal{B}(I), \gamma_n(A_1, ..., A_n) = \gamma_n(A_{\sigma_n(1)}, ..., A_{\sigma_n(n)})$$

is the consistency requirement indicating the surplus of a group is invariant under its members' ordering.

2.3 Stable group assignment

Following the notions in [24], [14], [8], we say a group assignment $\gamma \in \Gamma_{sym}$ is stable if it corresponds to an imputation $u : I \to \mathbb{R}$ such that

1. for all $N' \leq n \leq N$, for γ_n-a.e. $C = (i_1, ..., i_n) \in I^n$, $\sum_{j=1}^{n} u(i_j) \leq v(C)$
2. for all $N' \leq n \leq N$, $C = (i_1, ..., i_n) \in I^n$, $\sum_{j=1}^{n} u(i_j) \geq v(C)$

As a comparison, text such as [12] by Gilles denotes this imputation as core when the game v is supper-additive. Here, we study stable assignments, a generalized version of matching, as we wish to study not only the final allocation but also the group structures in equilibrium.

The first condition is the feasibility condition, requiring group members can in total gain no more than the surplus of a group if this group is formed. Since we only need this constraint on formed groups, the inequality in condition 1 holds almost everywhere. The second condition is the no-block condition, asserting no coalition will block the assignment by generating a higher surplus by generating a higher surplus.

Although agents of the same type might be assigned into different groups in different ways, we have the equal treatment property in the stable scenario since, as we shall see in latter proof, groups will only be formed in an efficient way.

2.4 Optimization problems

Finally, we state two related optimization problems. Firstly we state the welfare maximization problem, or Monge-Kantorovich problem:

$$\sup_{\gamma \in \Gamma_{sym}} \sum_{k=N'}^{N} \int_{I^k} v_k d\gamma_k$$

As we will see in the next section, when $N' = N$, the welfare maximization problem is reduced to the classical multi-marginal Monge-Kantorovich problem with an additional symmetry constraint.
Secondly, we state its dual problem, the utility minimization problem or dual Kantorovich problem:

$$\inf_{u \in \mathcal{U}} \int_I u d\mu$$

where $\mathcal{U} = \left\{ u \in L^1(I) : \forall N' \leq n \leq N, \forall C = (i_1, ..., i_n) \in I^n, \sum_{j=1}^{n} u(i_j) \geq v(C) \right\}$ is the set of imputations such that no groups will block.
There are three steps to show the non-emptiness of stable group assignment: Firstly, we show the existence of a maximizer solving the welfare maximization problem; Secondly, we show the maximization problem and minimization problem are the same by stating a duality theorem; Lastly, we show the stable group assignment coincides with the solution of minimization problem.
We will state these statements formally and prove them in the following section.

3 Stable assignments for groups of finite size

In this section, we study the case when group sizes are bounded by two natural numbers. We proceed by two steps: Firstly, we work on the case where only groups of size N could be formed. This case has a clear connection to the multi-marginal optimal transport problem. Then we study the general case.

3.1 Statements and proofs when group size is N

In this subsection, we focus on the case that only groups of maximized size can be formed. Therefore, we analyze with one additional assumption which will be dropped later.

(A4) $N' = N$. Or equivalently, v is a real-valued function on I^N.

In this case, the choice set of welfare maximization problem contains probability measures on I^N. Hence, in this subsection, the set of assignments is Γ_{sym} defined by:

$$\Gamma_{sym} = \{\gamma \in \mathbb{P}(I^N) : (\pi_1)_{\#}\gamma = \mu, \sigma_{\#}\gamma = \gamma, \forall \sigma \in O(N)\}$$

For the purpose of comparison to the traditional multi-marginal optimal transport theory, we define

$$\Gamma = \{\gamma \in \mathbb{P}(I^N) : (\pi_j)_{\#}\gamma = \mu, \forall 1 \leq j \leq N\}$$

Proposition 1. *For the cooperative game $((I, \mu), v, N', N)$, if v satisfies (A1)-(A4), there exists a $\gamma \in \Gamma_{sym}$ solving the welfare maximization problem*

$$\sup_{\gamma \in \Gamma_{sym}} \int_I^N v d\gamma$$

In Appendix A, we attach a more direct proof which does not use the conclusion of the traditional result. This direct proof inspires our proof for the case of measure bounds later. Next, we state and prove the duality theorem, which states that there is no gap between welfare maximization problem and utility minimization problem.

Proposition 2. *For the cooperative game $((I, \mu), v, N', N)$, if v satisfies (A1)-(A4),*

$$\sup_{\gamma \in \Gamma_{sym}} \int_{I^N} v d\gamma = N \inf_{u \in \mathcal{U}} \int_I u d\mu$$

where $\mathcal{U} = \left\{u \in L^1(I) : \sum_{j=1}^N u(i_j) \geq v(C), \forall C = (i_1, i_2, ..., i_N) \in I^N\right\}$ *and the infimum could be attained.*

Lastly, similar to [14], we state and prove the relationship between stable assignment and constrained utility minimization problem.

Proposition 3. *For the cooperative game $((I, \mu), v, N', N)$, if v satisfies (A1)-(A4), the corresponding imputations of stable assignment coincides with the solutions of the utility minimization problem.*

Thanks to these three results, we could state our main theorem with the presence of the assumption (A4):

Corollary. *For the cooperative game $((I, \mu), v, N', N)$, if v satisfies (A1)-(A4), the set of stable assignment is non-empty.*

3.2 Statements and proofs for the general case

In this part, we drop the assumption (A4). In this case, we recall the maximization problem is in the form

$$\sup_{\gamma \in \Gamma_{sym}} \sum_{k=N'}^{N} \int_{I^k} v_k d\gamma_k$$

The following lemma states its relationship with the special case we discussed in previous subsection:

Lemma 1.

$$\sup_{\gamma \in \Gamma_{sym}} \sum_{k=N'}^{N} \int_{I^k} v_k d\gamma_k = \sup_{(\mu_k)_{k=N'}^{N} \in K_{N',N}(\mu)} \sum_{k=N'}^{N} \sup_{\gamma_k \in \Gamma_{sym}(\mu_k)} \int_{I^K} v_k d\gamma_k$$

where $K_{N',N}(\mu)$ contains all N-tuples $(\mu_{N'}, ..., \mu_N) \in (\mathbb{P}(I))^{N-N'+1}$ where μ_k are positive measures on I such that $\sum_{k=N'}^{N} \mu_k = \mu$. Additionally, $\Gamma_{sym}(\mu_k) = \{\gamma_k \in M_+(I^k) : (\pi_1)_\# \gamma_k = \mu_k, (\sigma_k)_\# \gamma_k = \gamma_k, \forall \sigma_k \in O(k)\}$.[2]

This lemma states that maximizing over all group assignments is equivalent to a two-step maximization. Note agents are formed by $N - N' + 1$ categories based on the size of the groups they will be assigned to. We say a agent is in category k if this agent will be assigned to a group of size k. Let $\mu_k \leq \mu$ be the distribution of agents in category k. The two-step maximization is as follows: Firstly, we fix the distribution of agents in each category and choose the best way to assign agents for each category k into groups of size k, then choose the best way to divide the agents into these categories.
Due to Lemma 1, we could generalize Proposition 1 to the general case if we could assert the choice set $K_{N',N}(\mu)$ is compact and the functional is upper semi-continuous. Therefore, we state and prove the following two lemmas:

Lemma 2. $K_{N',N}(\mu) \subset (\mathbb{P}(I))^{N-N'+1}$ *is sequentially compact.*

Lemma 3. *For any $N' \leq k \leq N$, if μ_k be a positive measure on I^k, the functional on μ_k*

$$\sup_{\gamma \in \Gamma_{sym}(\mu_k)} \int_{I^K} v_k d\gamma$$

is weak- upper-semi continuous.*

Now we state our existence result.

Proposition 4. *For the cooperative game $((I, \mu), v, N', N)$, if v satisfies (A1)-(A3), there exists a $\gamma \in \Gamma_{sym}$ solving the welfare maximization problem.*

[2] Note Γ_{sym} and $\Gamma_{sym}(\mu_k)$ is not related to each other directly. Γ_{sym} is a subset of the product space $M_+(I^{N'}) \times ... \times M_+(I^N)$ while $\Gamma_{sym}(\mu_k)$ is a subset of $M_+(I^k)$.

In addition, we can state the duality result without assuming (A4):

Proposition 5. *For the cooperative game* $((I, \mu), v, N', N)$*, if* v *satisfies (A1)-(A3),*

$$\sup_{\gamma \in \Gamma_{sym}} \sum_{k=N'}^{N} \int_{I^k} v_k d\gamma_k = \sup_{(\mu_k) \in K_{N',N}(\mu)} \sum_{k=N'}^{N} \sup_{\gamma_k \in \Gamma_{sym}(\mu_k)} \int_{I^K} v_k d\gamma_k = \sup_{(\mu_k) \in K_{N',N}(\mu)} \sum_{k=N'}^{N} k \inf_{u_k \in \mathcal{U}_k} \int_I u d\mu_k$$

and the extremum on the right hand side could be attained.

Lastly we state the relationship between stable assignment and constrained utility minimization problem:

Proposition 6. *For the cooperative game* $((I, \mu), v, N', N)$*, if* v *satisfies (A1)-(A3), the corresponding imputations of stable assignment coincides with the solutions of the utility minimization problem.*

By above propositions, we can establish the existence of a stable assignment:

Theorem 1. *For the cooperative game* $((I, \mu), v, N', N)$*, if* v *satisfies (A1)-(A3), the set of stable assignment is non-empty.*

3.3 A counter example when groups size is countable

It is natural to ask if the above arguments and conclusions will hold in the limit case where the group size is countable. Unfortunately, the answer is no due to the example below:

Example 1. Let the agent space be $I = [0, 1]$ with the Euclidean metric and initial distribution is uniform in Lebesgue measure. Consider $v(C) = |C|^2$ for all finite subset $C \subset I$. As the surplus of any fixed-size group is constant, v satisfies (A1)-(A3). If there exists a integrable function u on I, u is essentially bounded. That is, for any $\epsilon > 0$, there exists $M > 0$ and a set $I' \subset I$ with measure greater than $1 - \epsilon$, u maps I' to $[-M, M]$. However, we need $\sum_{i \in C} u(i) \geq v(C) = |C|^2$ for all coalition C. By taking a finite subset of size greater than M within I', we have the contradiction. Thus, any large enough group will block the assignment.

3.4 Relationship with literature

As Theorem 1 does not assume the measure on agent space to be non-atomic, our model can be degenerated to a number of classical result on discrete agent space in the literature. In particular, when the measure concentrates on n discrete point and the group size is bounded by 2, it reduces to the existence result of stable roommate problem in [21]. Compare to [6], we impose a weaker assumption on the surplus function. Our solution concept does not require that each agent must be assigned to only one other agent for sure. Thus, we prove the existence of a probabilistic stable arrangement is always possible with weaker assumptions. If we impose more structure on surplus function (e.g. surplus of two agents of the same sex is zero and different sex is positive), it reduces to the existence results in [24], [14], [8]. Our notion of stable assignment is related to f-core defined by Kaneko and Wooders in [15]. We argued the existence in an accurate sense and relax some of their assumptions, such as continuity, compactness and closedness, thus providing a generalization of their results.

4 Model for groups of positive size

4.1 Environment

In this section, we study a cooperative game $((I, \mu), v, \epsilon, \epsilon')$ defined as follows. Let a Polish space I be the type space of agents and $\mu \in \mathbb{P}(I)$ be the distribution of agents' types.
Similar to the formulation in [4], a group is denoted by a positive measure on I that is weakly smaller than μ. We study the groups of size not larger than $\epsilon \in (0, 1]$. However, to guarantee the compactness of the choice set, we also need to restrict our attention to group of size with a positive lower bound $\epsilon' \in (0, 1]$. Clearly, $\epsilon \geq \epsilon'$ here. In particular, this inequality can be equality in which case only groups of size $\epsilon = \epsilon'$ could be formed. To keep our notations clean, we use $\mathcal{G}$, rather than $\mathcal{G}_{\epsilon',\epsilon}$, to denote the set of groups. Mathematically,

$$\mathcal{G} = \{\nu \in \mathcal{M}_+(I) : \nu \leq \mu, \epsilon' \leq \nu(I) \leq \epsilon\}$$

In particular, a group is denoted by an element $\nu \in \mathcal{G}$. Note, the space $\mathcal{M}_+(I)$ is metrizable, as, by Theorem 8.3.2 in [1], the total variation norm $\|\cdot\|_0$ generates the same topology as the weak topology on $\mathcal{M}_+(I)$.
The surplus function $v : \mathcal{G} \to \mathbb{R}$ is a real valued function on the set of groups satisfying,

(B1) v is upper semi-continuous in weak-* topology
(B2) v is bounded from above: there is a lower-semi continuous function $a \in L^1(I)$ such that $v(\nu) \leq \int_I a d\nu$, for all $\nu \in \mathcal{G}$.

Similar to the case of finite bounds, (B1) states the continuity requirement of the surplus function and (B2) guarantees the integrability of the surplus function.

4.2 Group Assignment

We define an assignment to be an element γ such that

1. $\gamma \in \mathcal{M}_+(\mathcal{G})$
2. $\forall A \in \mathcal{B}(I)$, $\int_{\mathcal{G}} \nu(A) d\gamma(\nu) = \mu(A)$

And we use $\Gamma_{\mathcal{G}}$ to denote the set of assignments containing all assignment γ.
Intuitively, $\gamma(\nu)$ is the number of copies group ν is formed according to γ. The first condition asserts γ could be interpreted as an assignment plan. Note in this case, an assignment is not a probability measure, since if it is, by taking $A = X$ in the second condition, we will obtain left hand side is less than ϵ but right hand side is 1. The second condition is the market clearing condition which states the mass that agents of types belonging to A is assigned by assignment γ is equal to the mass of agents of types belonging to A.
Note the set $\Gamma_{\mathcal{G}}$ is non-empty as a $\frac{2}{\epsilon+\epsilon'}$ point mass on the measure $\frac{\epsilon+\epsilon'}{2}\mu \in \mathcal{G}$ is an element in it.

4.3 Stable group assignment

Similar to the case when group sizes are finite, we say an assignment $\gamma \in \Gamma_{\mathcal{G}}$ is stable if there is an imputation $u : I \to \mathbb{R}$ such that

1. $\int_I u d\nu \leq v(\nu)$ γ-a.e.,
2. $\int_I u d\nu \geq v(\nu)$

The first condition is the feasibility condition requiring group members can in total gain no more than the surplus the group could get if they are assigned together. The second condition is the no-block condition, asserting no feasible group ν could block the assignment by generating a higher surplus.

4.4 Optimization Problems

We start by stating the welfare maximization problem, or Monge-Kantorovich problem:

$$\sup_{\gamma \in \Gamma_{\mathcal{G}}} \int_{\mathcal{G}} v d\gamma$$

Secondly, we state the utility minimization problem, or dual Kantorovich problem:

$$\inf_{u \in \mathcal{U}_{\mathcal{G}}} \int_I u d\mu$$

where $\mathcal{U}_{\mathcal{G}} = \left\{ u \in L^1(I) : \forall 1 \leq n \leq N, \int_I u d\nu \geq v(\nu), \forall \nu \in \mathcal{G} \right\}$ is the set of imputations such that no group has incentive to block.
We apply the same arguments to prove the existence of a stable assignment. There are three steps to show the non-emptiness of stable group assignment: Firstly, we show the existence of a maximizer solving the welfare maximization problem. Secondly, we show the welfare maximization problem and utility minimization problem have no gaps between each other by stating a duality theorem. Lastly, we show the set of corresponding imputations of stable group assignment coincides with the solutions of utility minimization problem.

5 Stable assignment of groups of positive size

In this section, we employ the same three-step argument to claim the existence of a stable assignment.

Proposition 7. *For the cooperative game* $((I,\mu), v, \epsilon', \epsilon)$*, if* v *satisfies (B1)-(B2), there exists a* $\gamma \in \Gamma_{\mathcal{G}}$ *solving the welfare maximization problem*

$$\sup_{\gamma \in \Gamma_{\mathcal{G}}} \int_{\mathcal{G}} v d\gamma$$

Now we state and prove the duality theorem.

Proposition 8. *For the cooperative game* $((I,\mu), v, \epsilon', \epsilon)$*, if* v *satisfies (B1)-(B2),*

$$\sup_{\gamma \in \Gamma_{\mathcal{G}}} \int_{\mathcal{G}} v d\gamma = \inf_{u \in \mathcal{U}} \int_I u d\mu$$

where $\mathcal{U} = \{u \in L^1(\mu) : \int_I u d\nu \geq v(\nu), \forall \nu \in \mathcal{G}\}$ *and the infimum could be attained.*

The proof is similar to the proof of 2-marginal case in [9].

Proposition 9. *For the cooperative game* $((I,\mu), v, \epsilon', \epsilon)$*, if* v *satisfies (B1)-(B2), the corresponding imputations of stable assignment coincides with the solutions of the utility minimization problem.*

By above propositions, we can establish the existence of a stable assignment:

Theorem 2. *For the cooperative game* $((I,\mu), v, \epsilon', \epsilon)$*, if* v *satisfies (B1)-(B2), the set of stable assignments is non-empty.*

5.1 Relationship with the literature

Schmeidler in [22] proved, in a Walrasian economy with a continuum of agents and a finite dimensional commodity space, core and epsilon-sized core coincide with each other.

However, in the cooperative game setting studied in this paper, this equivalence result is not always true. We recall the stable assignment solves:

$$\text{argmax} \left\{ \int_{\mathcal{G}} v d\gamma : \int_{\|\nu\| \in [\epsilon', \epsilon]} \nu d\gamma = \mu \right\}$$

In case v has the property that $v(s\nu) > sv(\nu)$ for all $\nu \in \mathcal{G}$ and $s > 1$, we have the optimal γ should concentrate on $\{\nu : \|\nu\| = \epsilon\}$. Thus, for different upper bound on group size ϵ, we have different set of stable assignments and different associated imputations. Besides, by the similar logic, in case $v(s\nu) < sv(\nu)$ for all $\nu \in \mathcal{G}$ and $s < 1$, optimal γ should concentrate on $\{\nu : \|\nu\| = \epsilon'\}$. Therefore, in this case, different lower bound on group size ϵ' will induce different set of stable assignments and different associated imputations.

6 Concluding remarks and future work

To sum up, in this paper, we studied assignment problem when there are finite or positive bounds requirement on group sizes. The finite bounds case is related to the concept f-core and the latter bound is related to concepts core and ϵ-sized core. We showed, under no assumptions of non-atomicness, compactness and with a weaker version of continuity, that a stable assignment exists. Our setting is more general than the exchange economy setting as we allow the surplus of a group to become lower when some new consumers come to join.

This paper relates assignment problem with group size bounds to optimal transport (Monge-Kantorovich) problem. In particular, this paper relates the finite size bounds case corresponds to a multi-marginal optimal transport problem and the positive size bounds case corresponds to a "continuum-" marginal optimal transport problem. By varying the size of agent space and group size bounds, this paper provides a unified framework and generalizes the existence results of Scarf in [21], Shapley and Shubik in[24], Gresky, Ostroy and Zame in [14], Chiaporri, McCann, Nesheim, Pass in [8] and[7]. Besides, it completes the discussion of Schmeilder in[22] in a more general cooperative game setting. In terms of optimization problems, more than linking the existence of stable assignments to two optimization problems, this paper proves an easy corollary of the existence and duality property for multi-marginal optimal transport problem when transport plans are assumed to be symmetric and new existence and duality results for the continuum-marginal optimal transport problem.

The benefit of optimal transport problem is mentioned by Chiaporri, McCann and Nesheim's [8] and surveyed more detailedly in Villani's book [25]. For the completeness, we list a few as future work directions. To begin with, under approperiate assumptions Monge solution coincides with Kantorovich solution in 2-marginal case. By establishing this equivalence, one can show the existence of a pure stable assignment instantly. Besides, stable assignment and its related imputation are well characterized in 2-marginal case. In addition, there are sufficient conditions on surplus function to guarantee the uniqueness of stable assignment in 2-marginal case. Lastly, the analytical form of optimal transport plan is known for some special case of transport cost (e.g. the transport cost is the dyadic distance function) and numerous numerical methods have been introduced for this problem. All these nice properties above are well-studied in the classical 2-marginal case, but are not completely-understood in multi-marginal case, and are left widely open in "continuum-" marginal case.

Acknowledgement

I thank Truman Bewley, Robert Anderson, Beth Allen, John Geanakopolos, Xiangliang Li, Felix Kubler, Larry Samuelson and participants of Yale theory seminar for the inspiring discussions. I also benefit from the comments from anonymous referees of IJCAI-17 workshop on algorithmic game theory. I am responsible for all remaining mistakes and typos.

References

1. Vladimir I Bogachev. *Measure theory (Vol. 2)*. Springer Science & Business Media, 2007.
2. Olga N Bondareva. Some applications of linear programming methods to the theory of cooperative games. *Problemy kibernetiki*, 10:119–139, 1963.
3. Guillaume Carlier and Bruno Nazaret. Optimal transportation for the determinant. *ESAIM: Control, Optimisation and Calculus of Variations*, 14(4):678–698, 2008.
4. Yeon-Koo Che, Jinwoo Kim, and Fuhito Kojima. Stable matching in large economies. *Manuscript, Stanford Univ*, 2013.
5. Alfred Galichon Chiappori, Pierre-André and Bernard Salanié. The roommate problem is more stable than you think. 2014.
6. Pierre-André Chiappori, Alfred Galichon, and Bernard Salanié. The roommate problem is more stable than you think. 2014.
7. Pierre-André Chiappori, Robert McCann, and Brendan Pass. Multidimensional matching. *Preprint arXiv:1604.05771*, 2016.
8. Pierre-André Chiappori, Robert J McCann, and Lars P Nesheim. Hedonic price equilibria, stable matching, and optimal transport: equivalence, topology, and uniqueness. *Economic Theory*, 42(2):317–354, 2010.
9. Richard M Dudley. *Real analysis and probability*, volume 74. Cambridge University Press, 2002.
10. Francis Ysidro Edgeworth. *Mathematical psychics: An essay on the application of mathematics to the moral sciences*, volume 10. Kegan Paul, 1881.
11. David Gale and Lloyd S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.
12. Robert P Gilles. *The cooperative game theory of networks and hierarchies*, volume 44. Springer Science & Business Media, 2010.
13. Donald B Gillies. Solutions to general non-zero-sum games. *Contributions to the Theory of Games*, 4(40):47–85, 1959.
14. Joseph M. Ostroy Gretsky, Neil E. and William R. Zame. The nonatomic assignment model. *Economic Theory*, 2(1):103–127, 1992.
15. Mamoru Kaneko and Myrna Holtz Wooders. Cores of partitioning games. *Mathematical Social Sciences*, 3(4):313–327, 1982.
16. Mamoru Kaneko and Myrna Holtz Wooders. The nonemptiness of thef-core of a game without side payments. *International Journal of Game Theory*, 25(2):245–258, 1996.
17. Hans G. Kellerer. Duality theorems for marginal problems. *Zeitschrift fr Wahrscheinlichkeitstheorie und verwandte Gebiete*, 67(4):399–432, 1984.
18. Brendan Pass. Uniqueness and monge solutions in the multimarginal optimal transportation problem. *SIAM Journal on Mathematical Analysis*, 43(6):2758–2775, 2011.
19. Brendan Pass. Optimal transportation with infinitely many marginals. *Journal of Functional Analysis*, 264(4):947–963, 2013.

20. Brendan Pass. Multi-marginal optimal transport: theory and applications. *ESAIM: Mathematical Modelling and Numerical Analysis*, 49(6):1771–1790, 2015.
21. Herbert E Scarf. The core of an n person game. *Econometrica: Journal of the Econometric Society*, pages 50–69, 1967.
22. David Schmeidler. A remark on the core of an atomless economy. *Econometrica (pre-1986)*, 40(3):579, 1972.
23. Lloyd S Shapley. On balanced sets and cores. *Naval Research Logistics (NRL)*, 14(4):453–460, 1967.
24. Lloyd S. Shapley and Martin Shubik. The assignment game i: The core. *International Journal of game theory*, 1(1):111–130, 1971.
25. Cédric Villani. *Optimal transport: old and new.* Vol. 338. Springer Science & Business Media, 2008.

A Alternative proof of Proposition 1

We modified the proof in [25] to incorporate the symmetric condition as well as multi-marginal condition. Due to the additional restriction on choice set Γ_{sym}, we need to reprove it is closed.

Proof. Recall $\mathcal{P} \subset \mathbb{P}(I^N)$ is tight if for any $\epsilon > 0$, there is a compact set K_ϵ such that $p(I^N - K_\epsilon) \leq \epsilon$ for all $p \in \mathcal{P}$. Since I is Polish space, by Ulam's theorem, $\{\mu\}$ is tight in I. i.e. There exists a compact set K_ϵ, such that $\mu(I - K_\epsilon) < \epsilon$. Hence, Γ_{sym} is tight in $\mathbb{P}(I^N)$ as for any $\gamma \in \Gamma_{sym}$,

$$\gamma(I^N - (K_\epsilon, K_\epsilon, ..., K_\epsilon)) \leq N\mu(I - K_\epsilon) \leq N\epsilon$$

By Prokhorov's theorem, Γ_{sym} has compact closure in weak* topology. Thus, to show Γ_{sym} is compact, we only need to show it is closed: Taking $\gamma_k \to \gamma$ in weak* topology for $\gamma_k \in \Gamma_{sym}$. Note as π_1 is continuous map, we have for any $f \in C_b(I^N)$, $f \circ \pi_1 \in C_b(I)$,

$$\begin{aligned} \int_I f d(\pi_1)_\# \gamma &= \int_{I^N} f \circ \pi_1 d\gamma = \lim_{n\to\infty} \int_{I^N} f \circ \pi_1 d\gamma_n \\ &= \lim_{n\to\infty} \int_I f d(\pi_1)_\# \gamma_n = \int_I f d\mu \end{aligned}$$

i.e. we have $(\pi_1)_\# \gamma = \mu$. Similarly, as $\sigma \in O(n)$ is a continuous map on I^N, we obtain $\sigma_\# \gamma = \mu$. Therefore, Γ_{sym} is closed thus compact.
Next we show $\int v d\gamma$ is upper semi-continuous with respect to the variable γ: Taking $\gamma_k \to \gamma$. Without loss of generality, we can assume v to be nonnegative as v is bounded from above by $\sum_{n=1}^N a_n$, which is lower-semi continuous, we can study $v - \sum_{n=1}^N a_n$, which is a non-positive upper semi-continuous function. By the non-positive upper semi-continuity, there is a decreasing sequence of v_l converging to v, where v_l is continuous. By Monotone convergence theorem,

$$\int v d\gamma = \lim_{l\to\infty} \int v_l d\gamma = \lim_{l\to\infty} \lim_{n\to\infty} \int v_l d\gamma_n \geq \limsup_{n\to\infty} \int v d\gamma_n$$

Finally, taking a sequence of $\gamma_n \in \Gamma_{sym}$ approaching the supremum of $\int v d\gamma$, by compactness, there is a subsequence converge to some measure in Γ_{sym}. As the set Γ_{sym} is closed, the limit measure is in this set. And this limit measure is the maximizer as the functional with respect to γ is upper semi-continuous. ■

B Proof of Proposition 1

Proof. By the traditional existence result of multi-marginal transport problem in [25], there is a $\gamma_0 \in \Gamma$ solving the maximization problem:

$$\max_{\gamma \in \Gamma} \int_{I^N} v d\gamma$$

Define

$$\tilde{\gamma} = \frac{1}{N!} \sum_{\sigma \in O(N)} \sigma_{\#}\gamma_0$$

We claim $\tilde{\gamma}$ is also a solution to the above problem. It is easy to check $\tilde{\gamma}$ is symmetric: for any test function $f \in C_b(I^N)$, any permutation $\sigma' \in O(n)$,

$$\begin{aligned}\int_{I^N} f d\sigma'_{\#}\tilde{\gamma} = \int_{I^N} f \circ \sigma' d\tilde{\gamma} &= \int_{I^N} \left(\frac{1}{N!} \sum_{\sigma \in O(N)} f \circ \sigma' \circ \sigma \right) d\gamma_0 \\ &= \int_{I^N} \left(\frac{1}{N!} \sum_{\sigma \in O(N)} f \circ \sigma \right) d\gamma_0 = \int_{I^N} f d\tilde{\gamma}\end{aligned}$$

i.e. $\tilde{\gamma} = \sigma'_{\#}\tilde{\gamma}$ for all $\sigma' \in O(N)$. That is, $\tilde{\gamma} \in \Gamma_{sym}$. Moreover, since v is symmetric, $\tilde{\gamma}$ induces the same total welfare as γ:

$$\begin{aligned}\int_{I^N} v d\tilde{\gamma} =& \frac{1}{N!} \sum_{\sigma \in O(N)} \int_{I^N} v d\sigma_{\#}\gamma_0 = \frac{1}{N!} \sum_{\sigma \in O(N)} \int_{I^N} v \circ \sigma d\gamma_0 \\ =& \frac{1}{N!} \sum_{\sigma \in O(N)} \int_{I^N} v d\gamma_0 = \int_{I^N} v d\gamma_0\end{aligned}$$

Note as $\tilde{\gamma} \in \Gamma_{sym}$, we have,

$$\sup_{\gamma \in \Gamma_{sym}} \int_{I^N} v d\gamma \geq \int_{I^N} v d\tilde{\gamma} = \max_{\gamma \in \Gamma} \int_{I^N} v d\gamma$$

The converse relation is obvious, as $\Gamma_{sym} \subset \Gamma$. Thus, we have the equality,

$$\sup_{\gamma \in \Gamma_{sym}} \int_{I^N} v d\gamma = \max_{\gamma \in \Gamma} \int_{I^N} v d\gamma$$

Hence, we know $\tilde{\gamma}$ is a solution for the problem. ■

C Proof of Proposition 2

Proof. By the proof in section 3.1 of [3], we have

$$\max_{\gamma \in \Gamma} \int_{I^N} v d\gamma = \min_{u_j \in \tilde{\mathcal{U}}} \sum_{j=1}^{N} \int_I u_j d\mu$$

for $\tilde{\mathcal{U}} = \left\{ (u_1, ..., u_N) \in (L^1(I))^N : \sum_{j=1}^N u_j(i_j) \geq v(C), \forall C = (i_1, i_2, ..., i_N) \in I^N \right\}$ and the infimum could be achieved. On the other hand, by the proof of Proposition 1, we have

$$\max_{\gamma \in \Gamma_{sym}} \int_{I^N} v d\gamma = \max_{\gamma \in \Gamma} \int_{I^N} v d\gamma$$

So it remains to show

$$\min_{u_j \in \tilde{\mathcal{U}}} \sum_{j=1}^{N} \int_I u_j d\mu = N \inf_{u \in \mathcal{U}} \int_I u d\mu$$

and the infimum on the right hand could be achieved.
Firstly, for any $m \in \mathbb{N}$, take $u^* \in \mathcal{U}$ such that

$$N \int_I u^* d\mu \leq N \inf_{u \in \mathcal{U}} \int_I u d\mu + \frac{1}{m}$$

Take $u_j^* = u^*$ for all $1 \leq j \leq N$. Therefore, as $u^* \in \mathcal{U}$, for any $1 \leq j \leq N$, we have $u_j \in L^1(I)$ and $\sum_{j=1}^N u_j^*(i_j) \geq v(C)$ for any $C = (i_1, ..., i_N) \in I^N$. Thus, $(u_j^*)_{j=1}^N \in \tilde{\mathcal{U}}$. By definition,

$$\min_{u_j \in \tilde{\mathcal{U}}} \sum_{j=1}^N \int_I u_j d\mu \leq \sum_{j=1}^N \int_I u_j^* d\mu = N \int_I u^* d\mu \leq N \inf_{u \in \mathcal{U}} \int_I u d\mu + \frac{1}{m}$$

Taking m goes to infinity,

$$\sup_{\gamma \in \Gamma_{sym}} \int_{I^N} v d\gamma \leq N \inf_{u \in \mathcal{U}} \int_I u d\mu$$

Conversely, take $(u_j^*)_{j=1}^N \in \tilde{\mathcal{U}}$ solving the left hand side, and define $u^* = \frac{1}{N} \sum_{n=1}^N u_n^*$. Then, for any $C = (i_1, ..., i_N) \in I^N$, we have,

$$\begin{aligned}
\sum_{j=1}^N u^*(i_j) &= \frac{1}{N} \sum_{j=1}^N \sum_{n=1}^N u_n^*(i_j) = \frac{1}{N} \sum_{j=1} \left[\sum_{k=0}^{N-1} u_j(i_{j+k}) \right] \\
&\geq \frac{1}{N} \left[v(i_1, ..., i_N) + v(i_2, ..., i_N, i_1) + ... + v(i_N, i_1, ..., i_{N-1}) \right] \\
&= v(C)
\end{aligned}$$

The last equality is by v is symmetric. Moreover, it is clear that $u^* \in L^1(I)$ as $\|u^*\|_{L^1} \leq \frac{1}{N} \sum_{j=1}^N \|u_j\|_{L^1} < \infty$. Hence, $u^* \in \mathcal{U}$, and it implies

$$\min_{u_j \in \tilde{\mathcal{U}}} \sum_{j=1}^N \int_I u_j d\mu = \sum_{j=1}^N \int_I u_j^* d\mu = N \int_I u^* d\mu \geq N \inf_{u \in \mathcal{U}} \int_I u d\mu$$

In conclusion,

$$\min_{u_j \in \tilde{\mathcal{U}}} \sum_{j=1}^N \int_I u_j d\mu = N \inf_{u \in \mathcal{U}} \int_I u d\mu$$

Moreover, $u^* = \frac{1}{N} \sum_{j=1}^N u_j^*$ is the solution of the utility minimization problem. ■

D Proof of Proposition 3

Proof. Firstly, we take a stable assignment γ. By definition, there exists an imputation $u \in L^1(I)$ such that $\sum_{j=1}^N u(i_j) \geq v(C)$ for all $C = (i_1, i_2, ..., i_N) \in I^N$, and $\sum_{j=1}^N u(i_j) = v(C)$, for γ-a.e. $C = (i_1, i_2, ..., i_N) \in I^N$. Then, we have

$$\int_{I^N} v d\gamma = \sum_{j=1}^N \int_{I^N} u(i_j) d\gamma = \sum_{j=1}^N \int_I u(i_j) d\mu(i_j) = N \int_I u d\mu$$

Now taking any $y \in \mathcal{U}$, we have that $\sum_{j=1}^{N} y(i_j) \geq v(C)$ for all $C = (i_1, i_2, ..., i_N) \in I^N$. Similarly,

$$\int_{I^N} v d\gamma \leq N \int_I y d\mu$$

Therefore, $\int_I u d\mu \leq \int_I y d\mu$. Consequently, u solves the utility minimization problem.
Conversely, if u solves the utility minimization problem. we have $\sum_{j=1}^{N} u(i_j) \geq v(C)$, for all groups $C = (i_1, i_2, ..., i_N) \in I^N$. Then, by the existence result stated in Proposition 1 and duality theorem stated in Proposition 2, there is a γ solving the maximization problem and $\int_{I^N} v d\gamma = \int_I u d\mu$. Thus, for γ-a.e. $C = (i_1, i_2, ..., i_N) \in I^N$, $\sum_{j=1}^{N} u(i_j) = v(C)$. Thus, u is a corresponding imputation and γ is stable. ■

E Proof of Lemma 1

Proof. To be clean, we take

$$L = \sup_{\gamma \in \Gamma_{sym}} \sum_{k=N'}^{N} \int_{I^k} v_k d\gamma_k$$

$$R = \sup_{(\mu_k)_{k=N'}^{N} \in K_{N',N}(\mu)} \sum_{k=N'}^{N} \sup_{\gamma_k \in \Gamma_{sym}(\mu_k)} \int_{I^K} v_k d\gamma_k$$

By definition of supremum, for any $m \in \mathbb{N}$, there is a $\gamma^{(m)} \in \Gamma_{sym}$ such that

$$\sum_{k=N'}^{N} \int_{I^k} v_k d\gamma_k^{(m)} + \frac{1}{m} > L$$

For each $N' \leq k \leq N$, define $\mu_k^{(m)} = (\pi_1)_{\#}\gamma_k^{(m)}$. It is straightforward to show that $\mu_k^{(m)}$ is a positive measure on I^k. By $\sum_{k=N'}^{N} (\pi_1)_{\#}\gamma_k^{(m)}(I) = \mu(I)$, we know $\sum_{k=N'}^{N} \mu_k^{(m)} = \mu$, which implies $(\mu_k^{(m)})_{k=N'}^{N} \in K_{N',N}(\mu)$. As $\gamma^{(m)} \in \Gamma_{sym}$, $\gamma_k^{(m)} \in \Gamma_{sym}(\mu_k^{(m)})$. Therefore, $\sum_{k=N'}^{N} \int_{I^k} v_k d\gamma_k^{(m)} \leq R$, which implies $R + \frac{1}{m} \geq L$. Take $m \to \infty$, we have $R \geq L$. Conversely, for any $m \in \mathbb{N}$, we take $(\mu_k^{(m)})_{k=N'}^{N} \in K_{N',N}(\mu)$ and $\gamma_k^{(m)} \in \Gamma_k(\mu_k^{(m)})$ such that $\int_{I^k} v_k d\gamma_k^{(m)} + \frac{1}{m} > R$. Now we claim $\gamma^{(m)} = (\gamma_{N'}^{(m)}, ..., \gamma_N^{(m)})$ is in Γ_{sym}. Firstly, it is clear each coordinate $\gamma_k^{(m)}$ is a positive measure on I^k. Moreover, $\sum_{k=N'}^{N} (\pi_1)_{\#}\gamma_k^{(m)} = \sum_{k=N'}^{N} \mu_k^{(m)} = \mu$. Lastly, we note the symmetry of $\gamma^{(m)}$ follows from the fact $\gamma_k^{(m)}$ is symmetric. Thus, we have $\sum_{k=N'}^{N} \int_{I^k} v_k d\gamma_k^{(m)} \leq L$ which implies $L > R + \frac{1}{m}$. Take $m \to \infty$, we have $L \geq R$. Hence, $L = R$. ■

F Proof of Lemma 2

Proof. We first show $K_N(\mu) := K_{1,N}(\mu) \subset (\mathcal{P}(I))^N$ is sequentially compact by induction. When $N = 2$, to show $\{(\mu_1, \mu_2) \in (\mathcal{M}_+(I))^2 :$

$\mu_1 + \mu_2 = \mu\}$ is sequentially compact, we only need to show $H = \{\mu_1 \in \mathcal{M}_+(I) : \mu_1 \le \mu\}$ is sequentially compact: for any sequence $(\mu_1^{(m)}, \mu_2^{(m)}) \in K_2(\mu)$, as $\mu_1^{(m)} \subset H$, $\mu_1^{(m)}$ has a convergent subsequence $\mu_1^{(m_k)}$, $\mu_2^{(m_k)} = \mu - \mu_1^{(m_k)}$ converges. Consequently, $(\mu_1^{(m_k)}, \mu_2^{(m_k)})$ is a convergent subsequence in $K_2(\mu)$.

To see H is sequentially compact: Since I is Polish space, by Ulam's theorem, $\{\mu\}$ is tight in I. i.e. for any $\epsilon > 0$, there exists a compact set K_ϵ, such that $\mu(I - K_\epsilon) < \epsilon$. Thus, for any $0 \le \mu_1 \le \mu$, $\mu_1(I - K_\epsilon) < \epsilon$. It implies the set $\{\mu_1 \in \mathcal{M}_+(I) : \mu_1 \le \mu\}$ is tight. On the other hand, the total variation of any μ_1 in this set is bounded by $\mu(X) = 1$. By Prokhorov theorem for measures (Theorem 8.6.2 in [1]), the set $\overline{\{\mu_1 \in \mathcal{M}_+(I) : \mu_1 \le \mu\}}$ is sequentially compact. It is easy to check the set itself is closed, so $\{\mu_1 \in \mathcal{M}_+(I) : \mu_1 \le \mu\}$ is sequentially compact.

Now, suppose for $N \in \mathbb{N}$, $K_N(\mu) \subset (\mathcal{P}(I))^N$ is sequentially compact. Then,

$$K_{N+1}(\mu) = \left\{((\mu_1, ..., \mu_N), \mu_{N+1}) \in \mathcal{M}_+(I)^{N+1} : \mu_1 + ... + \mu_{N+1} = \mu\right\}$$
$$\equiv \left\{((\mu_1, ..., \mu_N), \mu_{N+1}) \in (\mathcal{M}_+(I))^{N+1} : 0 \le \mu_{N+1} \le \mu, (\mu_1, ..., \mu_N) \in K_N(\mu - \mu_{N+1})\right\}$$

Due to the separability of I, the weak convergence on $M_+(I)$ is metrizable by Levy-Prokhorov metric d_P generalized by the Kantorovich-Rubinstein norm $\|\cdot\|_0$ (Theorem 8.3.2 in [1]). Define the product metric d on $(\mathcal{M}_+(I))^N$ by $d((x_1, ..., x_N), (y_1, ..., y_N)) = \sum_{k=1}^N d_P(x_k, y_k)$. Now take any sequence $(\mu_1^{(m)}, ..., \mu_{N+1}^{(m)})$ in $K_{N+1}(\mu)$. Note $\mu_{N+1}^{(m)} \subset H$, it has a convergent subsequence $\mu_{N+1}^{(m_k)}$. So without loss of generality, we assume $\mu_{N+1}^{(m)}$ converges to some μ_{N+1}^*. It is easy to see $\mu_{N+1}^* \in H$ as H is closed. Now we denote the projection of $(\mu_1^{(m)}, ..., \mu_N^{(m)})$ on $K_N(\mu - \mu_{N+1}^*)$ by $P_K(\mu_1^{(m)}, ..., \mu_N^{(m)})$, and its complement by $P_K^\perp(\mu_1^{(m)}, ..., \mu_N^{(m)})$. By induction hypothesis, $K_N(\mu - \mu_{N+1}^*)$ is sequentially compact, thus there is a subsequence (m_k) such that $P_K(\mu_1^{(m_k)}, ..., \mu_N^{(m_k)})$ converges. Note

$$(\mu_1^{(m_k)}, ..., \mu_N^{(m_k)}) = P_K(\mu_1^{(m_k)}, ..., \mu_N^{(m_k)}) + P_K^\perp(\mu_1^{(m_k)}, ..., \mu_N^{(m_k)})$$

Thus, to see $(\mu_1^{(m_k)}, ..., \mu_N^{(m_k)})$ is convergent, it suffices to show $P_K^\perp(\mu_1^{(m_k)}, ..., \mu_N^{(m_k)})$ converges to zero as $k \to \infty$, or, equivalently,$d((\mu_1^{(m_k)}, ..., \mu_N^{(m_k)}), K_N(\mu - \mu_{N+1}^*))$ converges to 0. Note, for any m_k, $\mu_1^{(m_k)} + ... + \mu_{N+1}^{(m_k)} = \mu$. So,

$$\sum_{i=1}^N \left(\mu_i^{(m_k)} + \frac{1}{N}\mu_{N+1}^{(m_k)} - \frac{1}{N}\mu_{N+1}^*\right) = \mu - \mu_{N+1}^*$$

As a result, $(\mu_1^{(m_k)} - \frac{1}{N}\mu_{N+1}^* + \frac{1}{N}\mu_{N+1}^{(m_k)}, ..., \mu_N^{(m_k)} - \frac{1}{N}\mu_{N+1}^* + \frac{1}{N}\mu_{N+1}^{(m_k)}) \in K_N(\mu - \mu_{N+1}^*)$. Therefore,

$$\begin{aligned}
&d((\mu_1^{(m_k)}, ..., \mu_N^{(m_k)}), K_N(\mu - \mu_{N+1}^*)) \\
&\le d((\mu_1^{(m_k)}, ..., \mu_N^{(m_k)}), (\mu_1^{(m_k)} - \frac{1}{N}\mu_{N+1}^* + \frac{1}{N}\mu_{N+1}^{(m_k)}, ..., \mu_N^{(m_k)} - \frac{1}{N}\mu_{N+1}^* + \frac{1}{N}\mu_{N+1}^{(m_k)})) \\
&\le N \times \frac{1}{N}\left\|\mu_{N+1}^{(m_k)} - \mu_{N+1}^*\right\|_0 \\
&\to 0
\end{aligned}$$

Thus, we obtain a convergent subsequence of $(\mu_1^{(m)}, ..., \mu_{N+1}^{(m)})$. That is, $K_{N+1}(\mu)$ is sequentially compact. Hence, we know $K_N(\mu) := K_{1,N}(\mu) \subset (\mathbb{P}(I))^N$ is sequentially compact.
Next, we notice $K_{N',N}(\mu)$ could be embedded into $K_N(\mu)$ by taking the first $N' - 1$ coordinate to be zero. Thus, as the embedding image forms a closed subset of $K_N(\mu)$, $K_{N',N}(\mu)$ is sequentially compact. ■

G Proof of Lemma 3

Proof. By Proposition 2, we have the duality

$$\sup_{\gamma \in \Gamma_{sym}(\mu_k)} \int_{I^k} v_k d\gamma = k \inf_{u \in \mathcal{U}_k} \int_I u d\mu_k$$

where $\mathcal{U}_k = \{u \in L^1(I) : \sum_{j=1}^k u(i_j) \geq v(C), \forall C = (i_1, i_2, ..., i_k) \in I^k\}$. So we only need to show the upper semi-continuity of infimum on the right hand side. Take $\mu^{(m)} \to \mu_k$, by the existence result in Proposition 2, we could take the $u^* \in L^1(I)$ minimizing $\inf_{u \in \mathcal{U}_k} \int_I u d\mu_k$. Then as continuous functions are dense in L^1, $\int_I u^* d\mu^{(m)} \to \int_I u^* d\mu_k$, which implies

$$\limsup_m \inf_{u \in \mathcal{U}_k} \int_I u d\mu^{(m)} \leq \inf_{u \in \mathcal{U}_k} \int_I u d\mu_k$$

■

H Proof of Proposition 4

Proof. By lemma 1, 2 and 3, the solution of welfare maximization problem could be attained. ■

I Proof of Proposition 5

Proof. The first equality is asserted by Proposition 2 and Lemma 1. The second equality is asserted by Lemma 1. The extremum could be achieved is asserted by Lemma 2 and Lemma 3. ■

J Proof of Proposition 6

Proof. We use Proposition 5:

$$\sup_{\gamma \in \Gamma_{sym}} \sum_{k=N'}^{N} \int_{I^k} v_k d\gamma_k = \sup_{(\mu_k) \in K_{N',N}(\mu)} \sum_{k=N'}^{N} \sup_{\gamma_k \in \Gamma_{sym}(\mu_k)} \int_{I^K} v_k d\gamma_k$$

$$= \sup_{(\mu_k) \in K_{N',N}(\mu)} \sum_{k=N'}^{N} k \inf_{u_k \in \mathcal{U}_k} \int_I u d\mu_k$$

Moreover, by Propositions 1, 2 and the fact that $K_{N',N}(\mu)$, the supremum and infimum are attained. Firstly, we take a stable assignment

$\gamma^* = (\gamma^*_{N'}, ..., \gamma^*_N)$. By definition, there exists a $u^* \in L^1(I)$ such that, for all $N' \le k \le N$, $\sum_{j=1}^k u^*(i_j) \ge v_k(C)$ for all $C = (i_1, i_2, ..., i_k) \in I^k$, and $\sum_{j=1}^k u^*(i_j) = v_k(C)$, for all γ^*_k-a.e. $C = (i_1, i_2, ..., i_k) \in I^k$. Thus, $u^* \in \mathcal{U}_k$ for all $N' \le k \le N$ and

$$\sum_{k=N'}^{N} \int_{I^k} v_k d\gamma^*_k = \sum_{k=N'}^{N} \int_{I^k} \sum_{j=1}^{k} u^*(i_j) d\gamma^*_k(i_1, ..., i_k) = \sum_{k=N'}^{N} k \int_I u^* d\mu^*_k$$

where $\mu^*_k = \gamma^*_k(I^k)$.
Now take any $u_k \in \mathcal{U}_k$, we have, $\sum_{j=1}^k u_k(i_j) \ge v_k(C)$ for all $C = (i_1, i_2, ..., i_k) \in I^k$ by definition. Then, we have $\sum_{k=N'}^N \int_{I^k} v_k d\gamma_k \le \sum_{k=N'}^N k \int_I u_k d\mu_k$ for any $\gamma \in \Gamma_{sym}$. Therefore,

$$\sum_{k=N'}^{N} k \int_I u^* d\mu^*_k \le \sum_{k=N'}^{N} k \int_I u_k d\mu^*_k$$

Hence,

$$\sum_{k=N'}^{N} k \int_I u^* d\mu^*_k \le \sum_{k=N'}^{N} k \inf_{u_k \in \mathcal{U}_k} \int_I u_k d\mu^*_k \le \sup_{(\mu_k)_{k=N'}^N \in K_{N',N}(\mu)} \sum_{k=N'}^{N} k \inf_{u_k \in \mathcal{U}_k} \int_I u_k d\mu_k$$

That is the sequence $(u_k = u^*, \mu_k = \mu^*_k)_{k=N'}^N$ solves the utility minimization problem.
Conversely, by the existence result in Proposition 5, we pick $(u^*_k, \mu^*_k)_{N' \le k \le N}$ solving the utility minimization problem. Define $u^* = \max_{N' \le k \le N} u^*_k$, then it is clear that for any $N' \le k \le N$, $C = (i_1, ..., i_k) \in I^k$, $\sum_{j=1}^k u^*(i_j) \ge \sum_{j=1}^k u^*_j(i_j) \ge v_k(C)$. Moreover, we note μ^*_k is supported by a subset of $\{i \in I : u^*_k(i) = u^*(i)\}$. By the duality in Proposition 5, for fixed μ^*_k's, there exists $\gamma_k \in \Gamma_{sym}(\mu^*_k)$ such that $\int_{I^k} v_k d\gamma^*_k = k \int_I u^*_k d\mu^*$. Thus, $\sum_{j=1}^k u^*_k(i_j) = v_k(C)$ for all γ^*_k-a.e. $C = (i_1, i_2, ..., i_k) \in I^k$. But we note γ^*_k has marginal μ_k, which is supported on $\{i \in I : u^*_k(i) = u^*(i)\}$, so γ^*_k is supported by a subset of the k-fold Cartesian product $\{i \in I : u^*_k(i) = u^*(i)\}^k$. Consequently, we have $\sum_{j=1}^k u^*_k(i_j) = v_k(C)$ for all γ^*_k-a.e. $C = (i_1, i_2, ..., i_k) \in \{i \in I : u^*_k(i) = u^*(i)\}^k$. But on $\{i \in I : u^*_k(i) = u^*(i)\}$, $u^*_k = u^*$, so, $\sum_{j=1}^k u^*_k(i_j) = v_k(C)$ for all γ^*_k-a.e. $C = (i_1, i_2, ..., i_k) \in I^k$. Hence, we showed $\gamma^* = (\gamma^*_{N'}, ..., \gamma^*_N)$ corresponds to an imputation u^*, thus is a stable assignment. ■

K Proof of Proposition 7

Proof. We proceed in two steps. Firstly, we show the choice set $\Gamma_{\mathcal{G}}$ is compact. Then, we show the functional with respect to γ is upper semi-continuous in weak-* topology.
Since I is a Polish space, by Ulam's theorem, $\{\mu\}$ is tight in I. i.e. For any $\epsilon_1 > 0$, there is a compact set $K \subset I$, such that $\mu(I - K) < \epsilon_1$. Hence, $\mathcal{G}$ is tight in $\mathcal{M}_+(I)$ as for any $\nu \in \mathcal{G}$, $\nu(I - K) \le \mu(I - K) \le \epsilon_1$. By Prokhorov theorem for measures (Theorem 8.6.2 in [1]), $\bar{\mathcal{G}}$ is sequentially

compact. It is easy to check $\mathcal{G}$ is closed and we have argued $\mathcal{G} \subset \mathcal{M}_+(\mu)$ has a metric, and a norm, we have $\mathcal{G}$ is compact.
Therefore, $\Gamma_{\mathcal{G}} \subset \mathcal{M}_+(\mathcal{G})$ is tight. On the other hand, as $\|\nu\|_0 \geq \epsilon'$, therefore, $\|\gamma\|_0 \leq 1/\epsilon'$ for all $\gamma \in \Gamma_{\mathcal{G}}$. Thus, again by Prokhorov theorem for measures, $\Gamma_{\mathcal{G}}$ is sequentially compact.
To see $\int_{\mathcal{G}} vd\gamma$ is upper-semi continuous on γ. Let $\gamma_n \to \gamma$ in $\mathcal{M}_+(\mathcal{G})$. We take a decreasing sequence of v_l converging to v pointwisely, where v_l is continuous. By Monotone convergence theorem and the fact v_l is decreasing,

$$\int_{\mathcal{G}} vd\gamma = \lim_{l\to\infty} \int_{\mathcal{G}} v_l d\gamma = \lim_{l\to\infty} \lim_{n\to\infty} \int_{\mathcal{G}} v_l d\gamma_n \geq \limsup_{n\to\infty} \int_{\mathcal{G}} vd\gamma_n$$

Finally, taking a sequence of $\gamma_n \in \Gamma_{sym}$ approaching the supremum of $\int_{\mathcal{G}} vd\gamma$, by compactness, there is a subsequence converging to some measure in Γ_{sym}. As the set Γ_{sym} is closed, this limit measure is also in Γ_{sym}. This limit measure is the maximizer since the functional $\int_{\mathcal{G}} vd\gamma$ with respect to γ is upper semi-continuous. ■

L Proof of Proposition 8

Proof. To begin with, for any $u \in \mathcal{U}_{\mathcal{G}}$, $\int_I ud\nu \geq v(\nu)$, which implies $\int_I ud\mu = \int_{\mathcal{G}} \int_I ud\nu d\gamma \geq \int_{\mathcal{G}} vd\gamma$ for any $\gamma \in \Gamma_{\mathcal{G}}$. Hence, $\sup_{\gamma\in\Gamma_{\mathcal{G}}} \int_{\mathcal{G}} vd\gamma \leq \inf_{u\in\mathcal{U}} \int_I ud\mu$. From now on, we prove its converse relation: For each $u \in C(I)$, define $F_u \in C(\mathcal{G})$ by

$$F_u(\nu) = \int_I ud\nu$$

The continuity of F_u is instant by definition of weak convergence. Let L be a space containing all such F_u, that is, $L = \{F_u : u \in C(I)\}$. To see L is a linear space, we note for any $a \in \mathbb{R}$, $aF_u = F_{au}$ and $F_u + F_{u'} = F_{u+u'}$. Next, define H to be a subset of $C(\mathcal{G})$ by $H = \{T \in C(\mathcal{G}) : T(\nu) \geq v(\nu)\}$. It is easy to see H is convex. As v is bounded from above by a continuous function, the interior of H is non-empty (The continuous function plus one is in interior). Now, we define a linear form r on L, by

$$r(F_u) = \int_I ud\mu$$

As $aF_u + bF_{u'} = F_{au+bu'}$ for any $u, u' \in C(I), a, b \in \mathbb{R}$, we know r is linear. Moreover, r is bounded below on $L \cap H$, since, for any $\gamma \in \Gamma_{\mathcal{G}}$ (we have noticed that the set is non-empty in the definition part),

$$r(F_u) = \int_I ud\mu = \int_{\mathcal{G}} \int_I ud\nu d\gamma \geq \int_{\mathcal{G}} v(\nu)d\gamma \geq \inf_{\nu\in\mathcal{G}} v(\nu)$$

Note as v is upper-semi continuous on a compact set, $\inf_{\nu\in\mathcal{G}} v(\nu) < \infty$, that is r is bounded from below. By Hahn-Banach theorem (Theorem 6.2.11 in [9]), r can be extended to a linear form $\tilde{r}$ on $C(\mathcal{G})$ such that,

$$\inf_{T\in H} \tilde{r}(T) = \inf_{T\in L\cap H} r(T)$$

Now we argue r is a positive functional: For any $T \geq 0$ in $C(\mathcal{G})$, we have $\tilde{v} + cT + 1 \in H$, where $\tilde{v}$ is a continuous function close enough to v, for any positive c. Note $\inf_H \tilde{r} = \inf_{H \cap L} r \geq \inf v$, by taking c large enough, we get $\tilde{r}(T) \geq 0$. To see r is bounded, we note for any $T \in C(\mathcal{G})$, $|r(T)| \leq |r(1)| \, \|T\|_{\infty}$.
Hence, by G is compact, by Riesz-Markov-Kakutani representation theorem (Theorem 7.4.1 in [9]), there exists a finite positive regular Borel measure ρ such that $\tilde{r}(T) = \int_{\mathcal{G}} T d\rho$ for any $T \in C(\mathcal{G})$. Lastly we show $\rho \in \Gamma_{\mathcal{G}}$. Note for $T \in L$, $T(\nu) = \int_I u d\nu$ if $T = F_u$. Therefore, for any continuous function u on $\mathcal{G}$,

$$\int_{\mathcal{G}} \int_I u d\nu d\rho = \int_I u d\mu$$

we know $\rho \in \Gamma_{\mathcal{G}}$. Therefore,

$$\inf_{u \in \mathcal{U}_{\mathcal{G}}} \int_I u d\mu = \inf_{T \in H \cap L} r(T) = \inf_{T \in H} \tilde{r}(T) = \inf_{T \in H} \int_{\mathcal{G}} T d\rho = \int_{\mathcal{G}} v d\rho \leq \sup_{\gamma \in \Gamma_{\mathcal{G}}} \int_{\mathcal{G}} v d\gamma$$

On the other hand, the infimum could be attained follows from the proof in [17], as a consequence of uniform integrability. ■

M Proof of Proposition 9

Proof. Firstly, we take a stable assignment γ. By definition, there exists a $u \in L^1(I)$ such that $\int_I u d\nu \geq v(\nu)$ for all $\nu \in \mathcal{G}$, and $\int_I u d\nu \leq v(\nu)$, for all γ-a.e. ν. Therefore, $\int_{\mathcal{G}} v d\gamma = \int_{\mathcal{G}} \int_I u d\nu d\gamma = \int_I u d\mu$ Now taking any $y \in \mathcal{U}_{\mathcal{G}}$, we have that $\int_I y d\nu \geq v(\nu)$ for all $\nu \in \mathcal{G}$. Thus, $\int_{\mathcal{G}} v d\gamma \leq \int_{\mathcal{G}} \int_I y d\nu d\gamma = \int_I y d\mu$. Therefore, $\int_I u d\mu \leq \int_I y d\mu$. Thus, u solves the minimization problem.
Conversely, if u solves the minimization problem. we have $\int_I u d\nu \geq v(\nu)$, for all groups $\nu \in \mathcal{G}$. Then, by the duality theorem Proposition 8, there is a γ solving the welfare maximization problem and $\int_I v d\gamma = \int_I u d\mu$. Thus, for γ-a.e. ν, $\int_I u d\nu = v(\nu)$. As a result, u is a corresponding imputation and γ is stable. ■